Stubborn observers for linear and nonlinear ~~uncertain~~ systems

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Outline

Stubborn Luenberger Observers for Linear Time-Invariant Plants

- Continous-time results
- Discrete-time results
- Examples

Stubborn Redesign for Nonlinear High-Gain Observers

Example

Stubborn Luenberger Observers for Linear Time-Invariant Plants

Dynamically saturated linear output injection

- Plant equations:

$$\begin{cases} \dot{x} = Ax + B_u u \\ y = Cx + D_u u \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$, and $y \in \mathbb{R}^{n_y}$.

- Observer equations:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_u u + K \, \mathrm{sat}_{\sigma}(y - \hat{y}) \\ \hat{y} &= C\hat{x} + D_u u \end{aligned}$$

with $\sigma = \begin{bmatrix} \sigma_1 & \cdots & \sigma_{n_y} \end{bmatrix}^T \in \mathbb{R}^{n_y}$ symmetric saturation levels

- Adaptive saturation law:

$$\begin{aligned} \dot{\overline{\sigma}} &= -\lambda \overline{\sigma} + (y - \hat{y})^\top R \, (y - \hat{y}), \quad \overline{\sigma} \in \mathbb{R}_{\geq 0} \\ \sigma_i &= \sqrt{\overline{\sigma}/w_i}, \quad i = 1, \ldots, n_y, \end{aligned}$$

with constants $\lambda > 0$, $R = R^\top > 0$ and $w_i > 0$, to be chosen.

Error dynamics and underlying philosophy

- Estimation error: $e \triangleq x - \hat{x}$ ($e_y = Ce = y - \hat{y}$) obeys dynamics:

$$\begin{aligned} \dot{e} &= Ae - K\,\mathrm{sat}_{\sigma}(Ce) \\ \dot{\overline{\sigma}} &= -\lambda\overline{\sigma} + e^T C^T R C e, \quad \overline{\sigma} \in \mathbb{R}_{\geq 0} \end{aligned}$$

- Set $\mathbb{R}_{\geq 0}$ is forward invariant for the dynamics of $\overline{\sigma}$, then $\sigma_i = \sqrt{\overline{\sigma}/w_i}$ is well defined.
- The observer is "stubborn": the output estimation error $e_y = y - \hat{y} = Ce$ is injected through a saturation $\Rightarrow$ measurement outliers are saturated to reduce their effect on the error e.
- Dynamic adaptation of the saturation level is necessary to get global asymptotic stability of the estimation error.
- Dynamics is described by $\dot{\xi} = f(\xi),\ x \in \mathcal{C}$ with f continuous and $\mathcal{C}$ closed

Error dynamics provides intuitive evolution

- Error dynamics depends on green parameters

$$\begin{aligned} \dot{e} &= Ae - K\,\mathrm{sat}_\sigma(Ce) \\ \dot{\overline{\sigma}} &= -\lambda\overline{\sigma} + e^T C^T R C e, \quad \overline{\sigma} \in \mathbb{R}_{\geq 0} \end{aligned}$$

- Parameter K is the standard Luenberger gain
- Parameter $\lambda > 0$ forces the saturation level σ to converge exponentially to zero if $e_y = Ce = 0$
- Parameter $R > 0$ causes necessary increase of σ wen $e_y \neq 0$ so that the error dynamics can be stabilized.
- A fourth parameter diagonal positive definite $W = W^T > 0$ comprising $W = \mathrm{diag}\{w_i\}$ appears in $\sigma_i = \sqrt{\overline{\sigma}/w_i}$
- **Full-stubborn design involves the selection of K, λ, R, W.**
- **Stubborn redesign involves the selection of λ, R, W.** only

GES of error dynamics uses sector conditions

- Equivalent expression of error dynamics:

$$\begin{aligned} \dot{e} &= (A - KC)e + K(\overbrace{Ce - \mathrm{sat}_{\sigma}(Ce)}^{:=\mathrm{dz}_{\sigma}(e_y)}) \\ \dot{\overline{\sigma}} &= -\lambda\overline{\sigma} + e^T C^T R C e, \quad \overline{\sigma} \in \mathbb{R}_{\geq 0} \end{aligned}$$

- Global sector condition (holds for any e, any diagonal $U > 0$)

$$\mathrm{dz}_{\sigma}(e_y)^T U(Ce - \mathrm{dz}_{\sigma}(e_y)) \geq 0$$

- Regional sector condition (holds only for some e, any diagonal $W > 0$)

$$\mathrm{dz}_{\sigma}(He) = 0 \;\Rightarrow\; \mathrm{dz}_{\sigma}(e_y)^T W(Ce + He - \mathrm{dz}_{\sigma}(e_y)) \geq 0,$$

- Stability is proven using a nonsmooth Lyapunov function

$$V(e,\overline{\sigma}) = \underbrace{e^T P e + \zeta\overline{\sigma}}_{V_{REG}(e,\overline{\sigma})} + \eta \underbrace{\max\{e^T P e - \overline{\sigma}, 0\}}_{V_{GLOB}(e,\overline{\sigma})},$$

Main stability theorem with design LMIs

Theorem. Consider any feasible solution to the matrix inequalities

$$\mathrm{He}\begin{bmatrix} PA - XC + \frac{1}{2}(\lambda P - C^T RC) & X \\ UC & -U \end{bmatrix} < 0. \qquad (1)$$

$$\mathrm{He}\begin{bmatrix} PA - XC & X \\ WC + Y & -W \end{bmatrix} < 0 \qquad (2)$$

$$\begin{bmatrix} P & Y_i^T \\ Y_i & w_i \end{bmatrix} \geq 0, \quad i = 1, \dots, n_y, \qquad (3)$$

(where $\mathrm{He}(Z) = Z + Z^T$)
and select $K = P^{-1}X$. Then the error dynamics is GES

Proof Sketch: $V(e, \overline{\sigma}) = e^T Pe + \zeta\overline{\sigma} + \eta \max\{e^T Pe - \overline{\sigma}, 0\}$

- Use V_{REG} and regional sector in $e^T Pe - \overline{\sigma} \leq 0$ (where $V_{GLOB} = 0$ and $\mathrm{dz}_\sigma(He) = 0$) and fix ζ
- Use V_{GLOB} and global sector in $e^T Pe - \overline{\sigma} \geq 0$ and fix η
- both steps prove $\dot{V}(e, \overline{\sigma}) \leq -c_3(|e|^2 + |\sigma|^2)$

Feasibility and optimization

Proposition: LMIs (1), (2), (3) are feasible if and only if pair (C, A) is detectable

Optimized parameter selection: can be performed by

- First solving LMIs maximizing λ (want σ to go fast to zero)
- Then fix λ at a fraction of its maximum and
  1. fix $W > I$ and $P > I$ for improved numerical conditioning (LMIs are homogeneous)
  2. minimize the size of matrix R to reduce overshoot of σ or
  3. minimize the size of $K = P^{-1}X$ by indirectly minimizing the size of X or
  4. minimize the $\mathcal{L}_2$ gain from a disturbance to the estimation error (bounded real lemma)

Alternative optimization criteria may be proposed (trade-offs?)

Discrete-time results follow parallel formulation

- Plant equations:

$$\begin{cases} x^{+} = Ax + B_u u \\ y = Cx + D_u u \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$, and $y \in \mathbb{R}^{n_y}$.

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with constants $\lambda > 0$, $R = R^\top > 0$ and $w_i > 0$, to be chosen.

Error dynamics and underlying philosophy

- Estimation error: $e \triangleq x - \hat{x}$ ($e_y = Ce = y - \hat{y}$) obeys dynamics:

$$
\begin{aligned}
e^+ &= Ae - L\,\text{sat}_\sigma(Ce) \\
\overline{\sigma}^+ &= \lambda\overline{\sigma} + e^T C^T RCe, \quad \overline{\sigma} \in \mathbb{R}_{\geq 0}
\end{aligned}
$$

- Set $\mathbb{R}_{\geq 0}$ is forward invariant for the dynamics of $\overline{\sigma}$, then $\sigma_i = \sqrt{\overline{\sigma}/w_i}$ is well defined.
- The observer is "stubborn": the output estimation error $e_y = y - \hat{y} = Ce$ is injected through a saturation $\Rightarrow$ measurement outliers are saturated to reduce their effect on the error e.
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Main stability theorem with design LMIs

Theorem. Consider any feasible solution to the matrix inequalities

$$\text{He}\begin{bmatrix} -\frac{1}{2}(\lambda P + C^\top R C) & 0 & 0 \\ UC & -U & 0 \\ PA - XC & X & -\frac{1}{2}P \end{bmatrix} < 0. \tag{4}$$

$$\text{He}\begin{bmatrix} -\frac{1}{2}P & 0 & 0 \\ WC + Y & -W & 0 \\ PA - XC & X & -\frac{1}{2}P \end{bmatrix} < 0 \tag{5}$$

$$\begin{bmatrix} P & Y_i^T \\ Y_i & w_i \end{bmatrix} \geq 0, \quad i = 1, \ldots, n_y, \tag{6}$$

and select $K = P^{-1}X$. Then the error dynamics is GAS

Proof Sketch (wrong!): $V(e, \overline{\sigma}) = e^T Pe + \zeta\overline{\sigma} + \eta \max\{e^T Pe - \overline{\sigma}, 0\}$

- Use V_{REG} and regional sector in $e^T Pe - \overline{\sigma} \leq 0$ (where $V_{GLOB} = 0$ and $\text{dz}_\sigma(He) = 0$) and fix ζ
- Use V_{GLOB} and global sector in $e^T Pe - \overline{\sigma} \geq 0$ and fix η
- both steps prove $\dot{V}(e, \overline{\sigma}) \leq -c_3(|e|^2 + |\sigma|^2)$

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$$\mathrm{He}\begin{bmatrix} -\frac{1}{2}(\lambda P + C^\top R C) & 0 & 0 \\ UC & -U & 0 \\ PA - XC & X & -\frac{1}{2}P \end{bmatrix} < 0. \tag{7}$$

$$\mathrm{He}\begin{bmatrix} -\frac{1}{2}P & 0 & 0 \\ WC + Y & -W & 0 \\ PA - XC & X & -\frac{1}{2}P \end{bmatrix} < 0 \tag{8}$$

$$\begin{bmatrix} P & Y_i^T \\ Y_i & w_i \end{bmatrix} \geq 0, \quad i = 1, \dots, n_y, \tag{9}$$

and select $K = P^{-1}X$. Then the error dynamics is GAS

Proof Sketch:

- Prove UGAS of the set $\mathcal{E} := \{e^T Pe - \overline{\sigma} \leq 0\}$ (where $\mathrm{dz}_\sigma(He) = 0$)
- Prove stability of the origin from $\mathcal{E}$ and boundedness of solutions
- Apply results on cascaded nonlinear systems

Example 1: Cont. time 2nd-order marginally stable

- Continuous-time plant with:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

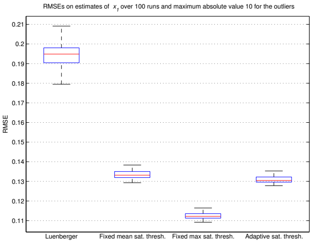
  and $u(t) = \sin(2\pi t)$.
- Design results:

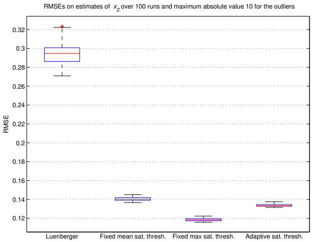
$$K = \begin{pmatrix} -11.9944 \\ 20.4153 \end{pmatrix} \quad R = 3.3256 \cdot 10^3 \quad W = 528.2705\,.$$

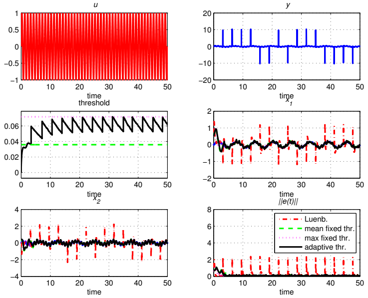
Table : Medians of the RMSEs on the estimates of x_1 and x_2 (max. abs. 10).

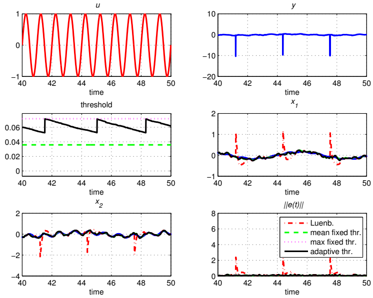
	outlier maximum absolute value									
	0.1		1		10		100		1000	
	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2
Luenberger obs.	0.11	0.12	0.11	0.13	0.19	0.29	1.65	2.71	18.01	29.97
Mean constant sat. obs.	0.16	0.16	0.16	0.16	0.13	0.14	0.12	0.13	0.13	0.17
Max. constant sat. obs.	0.14	0.15	0.14	0.15	0.11	0.12	0.12	0.13	0.17	0.26
Adapt. thresh. sat. obs.	0.13	0.13	0.13	0.13	0.13	0.13	0.14	0.15	0.18	0.23

Example 1: RMSE boxplots and simulation run









Ex. 2: 2nd-order marginally stable, & destabilizing u

- Continuous-time plant with:

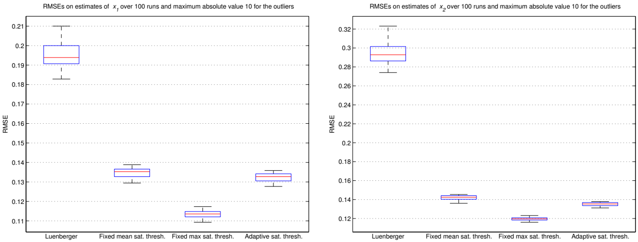
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

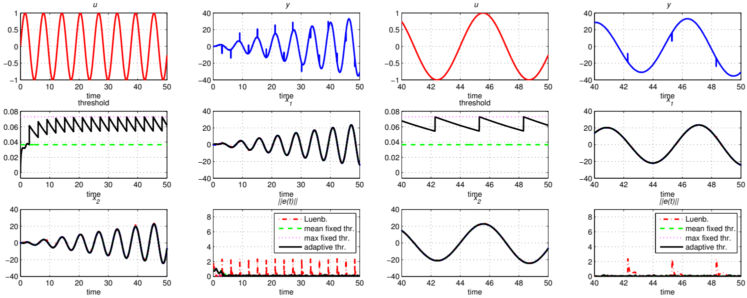
  as for Example 1 but with $u(t) = \sin(t)$.
- Design results: of course, the same of Example 1.

Table : Medians of the RMSEs on the estimates of x_1 and x_2 (max. abs. 10).

	outlier maximum absolute value									
	0.1		1		10		100		1000	
	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2
Luenberger obs.	0.11	0.12	0.11	0.13	0.19	0.29	1.63	2.68	16.39	26.96
Mean constant sat. obs.	0.16	0.16	0.16	0.16	0.14	0.14	0.12	0.12	0.13	0.16
Max. constant sat. obs.	0.14	0.15	0.14	0.15	0.11	0.12	0.12	0.13	0.16	0.24
Adapt. thresh. sat. obs.	0.13	0.13	0.13	0.13	0.13	0.14	0.14	0.15	0.17	0.22

Example 2: RMSE boxplots and simulation run





Example 3: second-order unstable system

- Continuous-time plant with:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0.1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

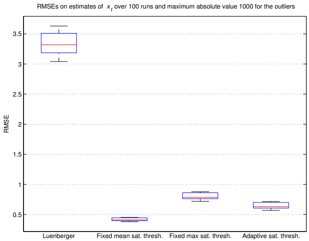
  and $u(t) = \sin(t)$
- Design results:

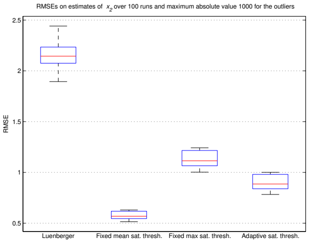
$$K = \begin{pmatrix} -6.7510 \\ 11.7510 \end{pmatrix} \quad R = 3.0895 \cdot 10^3 \quad W = 512.6717\,.$$

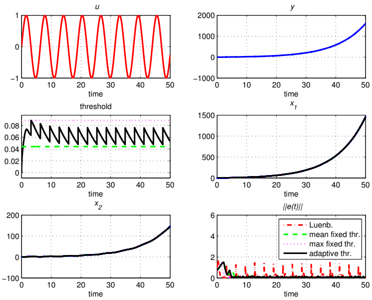
Table : Medians of the RMSEs on the estimates of x_1 and x_2 (max. abs. 10).

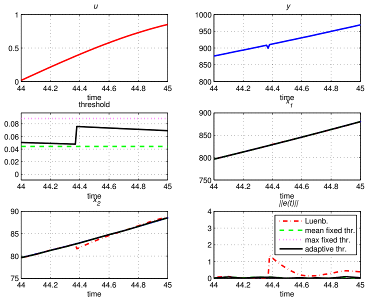
	outlier maximum absolute value									
	0.1		1		10		100		1000	
	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2
Luenberger obs.	0.13	0.13	0.13	0.13	0.18	0.24	1.29	2.0	13.01	20.34
Mean constant sat. obs.	0.29	0.17	0.29	0.17	0.25	0.16	0.15	0.14	0.14	0.16
Max. constant sat. obs.	0.19	0.14	0.14	0.12	0.18	0.14	0.14	0.14	0.17	0.21
Adapt. thresh. sat. obs.	0.25	0.17	0.15	0.13	0.25	0.17	0.25	0.18	0.26	0.22

Example 3: RMSE boxplots and simulation run









Example 4: Discrete-time sampled-data GPS

Sampled kinematic model of a mass point moving on a plane

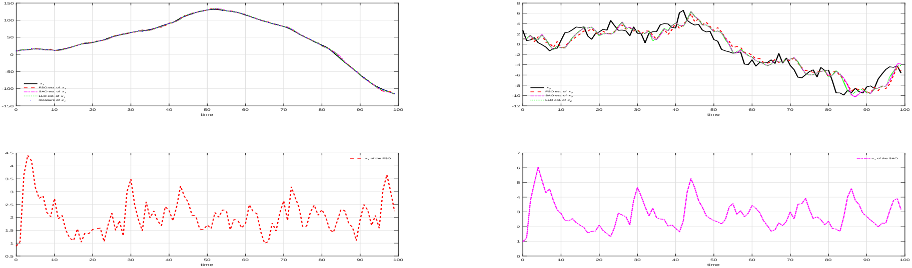
Disturbance d acts on dynamics and ν is measurements noise (with outliers)

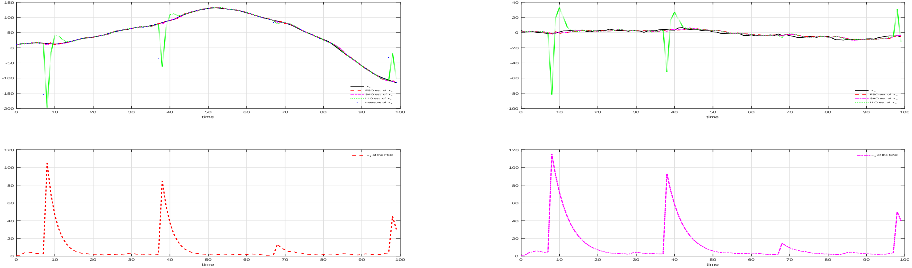
$$\left[\begin{array}{c|c} A & B_d \\ \hline C & D_d \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Compare the following three cases:

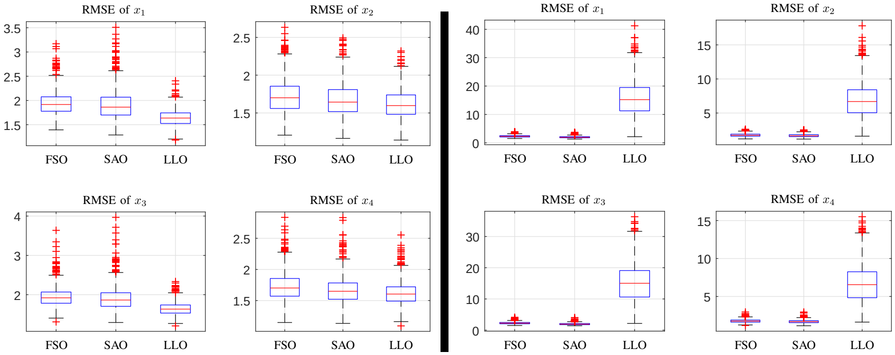
- Simple Luenberger (LL) with gain minimizing disturbance effect
- Stubborn Augmentation (SA) using the same Luenberger gain
- Full Stubborn (FS)

Example 4: Estimate and σ without and with outliers





Example 4: RMS of the error without and with outliers



The performance is slightly deteriorated in the absence of outliers (left) but greatly improved with outliers (right)

Stubborn Redesign for Nonlinear High-Gain Observers

Highlights on High-Gain Observers

- Plant equations:

$$\begin{cases} \dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_i, u) \qquad i = 1, \ldots, n-1 \\ \dot{x}_n &= \varphi_n(x_1, \ldots, x_n, u, d) \\ y &= x_1 + \nu \end{cases}$$

  with $x = \mathrm{col}(x_1, \ldots, x_n) \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^{n_u}$, and $y \in \mathbb{R}$, $\nu \in \mathbb{R}$ is some measurement noise, $d \in \mathbb{R}^{n_d}$ is some unknown disturbance and φ_i locally Lipschitz functions.
- High-Gain Observer equations:

$$\begin{cases} \dot{\hat{x}}_i &= \hat{x}_{i+1} + \hat{\varphi}_i(\hat{x}_1, \ldots, \hat{x}_i, u) + k_i\, \ell^i (y - \hat{y}) \qquad i = 1, \ldots, n-1 \\ \dot{\hat{x}}_n &= \hat{\varphi}_n(\hat{x}_1, \ldots, \hat{x}_n, u) + k_n\, \ell^n (y - \hat{y}) \\ \hat{y} &= \hat{x}_1 \end{cases}$$

  with $\hat{\varphi}_i(x, u) = \varphi_i(x, u)$ for all $(x, u) \in X \times U$ and bounded outside,

$$\lambda^n + k_1 \lambda^{n-1} + \ldots + k_{n-1}\lambda + k_n$$

  is Hurwitz and $\ell \geq 1$ large enough (high-gain parameter)

Functions φ_i and their relation to uncertainty

Functions φ satisfy the following conditions:

$$\begin{aligned}\hat{\varphi}_i(x_1,\ldots,x_i,u) &= \varphi_i(x_1,\ldots,x_i,u), \qquad i=1,\ldots,n-1,\\ \hat{\varphi}_n(x_1,\ldots,x_n,u) &= \varphi_n(x_1,\ldots,x_n,u,0),\end{aligned}$$

for all $(x,u) \in X \times U$ and

$$|\hat{\varphi}_i(\hat{x}_1,\ldots,\hat{x}_i,u) - \varphi_i(x_1,\ldots,x_n,u)| \le L_i \sum_{j=1}^{i} |\hat{x}_j - x_j|, i=1,\ldots,n-1$$

$$|\hat{\varphi}_n(\hat{x}_1,\ldots,\hat{x}_n,u) - \varphi_n(x_1,\ldots,x_n,u,d)| \le L_n \sum_{j=1}^{n} |\hat{x}_j - x_j| + R|d|$$

for all $(x,u,\hat{x}) \in X \times U \times \mathbb{R}^n$ and d

Generalizations are possible, to also consider **uncertainty in** φ_i

Scaled Error dynamics has Desirable Power Structure

- Estimation error: $e_i \stackrel{\triangle}{=} \ell^{-(i-1)}(x_i - \hat{x}_i)$ obeys dynamics:

$$\dot{e} \quad = \quad \ell(A - KC)e + \Delta_\varphi(x, e, u) + \ell K \nu$$

  where we used the compact notation $K = (k_1, \ldots, k_n)^T$, $C = (1, 0, \ldots, 0)$

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 0 & & & 0 \end{pmatrix}, \quad \Delta_\varphi(x, \hat{x}, u) = \begin{pmatrix} \vdots \\ \dfrac{\varphi_i(x, u) - \hat{\varphi}_i(\hat{x}, u)}{\ell^{i-1}} \\ \vdots \end{pmatrix},$$

  and $|\Delta_\varphi(x, \hat{x}, u)| \leq L|e| + \ell^{-(n-1)} L|d|$ for some $L > 0$.
- We can prove convergence of the observer for ℓ large enough by choosing the Lyapunov function $V = e^T P e$ where $P = P^T > 0$ is solution of $PA + A^T P = -I$. In particular, convergence in ensured for $\boldsymbol{\ell > \ell^* = 2L|P|}$.

Main properties of high-gain observers

For $\ell > \ell^*$ we can show that the following bound holds

$$|\hat{x}_i(t) - x_i(t)| \leq \alpha_1 \ell^{i-1} \exp(-\alpha_2 \ell\, t)|\hat{x}(0) - x(0)| + \frac{\alpha_3 |d|_\infty}{\ell^{n-(i-1)}} + \alpha_4 \ell^{i-1} |\nu|_\infty$$

- ✓ Arbitrarily fast exponential convergence of the estimation error for $d = 0$ and $\nu = 0$
- ✓ Robustness properties w.r.t. disturbances d with tunable ISS gain
- ✓ ISS w.r.t. measurement noise
- ✗ peaking phenomenon proportional to ℓ^{i-1}
- ✗ ISS-gain proportional to ℓ^{i-1} *(problem in presence of outliers)*

Stubborn ISS Redesign for High-Gain Observers

- Plant equations :

$$\begin{cases} \dot{x}_i = x_{i+1} + \varphi_i(x_1, \ldots, x_i, u) & i = 1, \ldots, n-1 \\ \dot{x}_n = \varphi_n(x_1, \ldots, x_n, d, u) \\ y = x_1 \end{cases}$$

- High-Gain Observer equations:

$$\begin{cases} \dot{\hat{x}}_i = \hat{x}_{i+1} + \hat{\varphi}_i(\hat{x}_1, \ldots, \hat{x}_i, u) + k_i\, \ell^i\, \mathrm{sat}_{\bar{\sigma}}(y - \hat{y}) \\ \qquad\qquad i = 1, \ldots, n-1 \\ \dot{\hat{x}}_n = \hat{\varphi}_n(\hat{x}_1, \ldots, \hat{x}_n, 0, u) + k_n\, \ell^n\, \mathrm{sat}_{\bar{\sigma}}(y - \hat{y}) \\ \hat{y} = \hat{x}_1 \end{cases}$$

- Adaptive saturation law:

$$\begin{cases} \dot{\sigma} = -\ell\lambda\sigma + \ell(\sigma + \epsilon)(y - \hat{y})^2, & \sigma \in \mathbb{R}_{\geq 0} \\ \bar{\sigma} = \sqrt{\sigma} \end{cases}$$

with constants $\lambda > 0$, $\epsilon > 0$ to be chosen.

Error dynamics and underlying philosophy

- Estimation error $e_i \triangleq \frac{x_i - \hat{x}_i}{\ell^{i-1}}$ obeys dynamics:

$$\begin{aligned} \dot{e} &= \ell A e + \Delta_\varphi(x, e, u) - \ell K \operatorname{sat}_{\bar{\sigma}}(Ce) \\ \dot{\sigma} &= -\ell\lambda\sigma + \ell(\lambda + \epsilon)e^T C^T C e, \qquad \sigma \in \mathbb{R}_{\geq 0} \end{aligned}$$

- Set $\mathbb{R}_{\geq 0}$ is forward invariant for the dynamics of $\overline{\sigma}$, then $\bar{\sigma} = \sqrt{\sigma}$ is well defined.
- The observer is "stubborn" in that the output estimation error $e_y = y - \hat{y} = Ce$ is injected in the observer equations through a bounded function and hence the outliers in the measurements are saturated so as to reduce their effect on the estimation error.
- Dynamic adaptation of the saturation level is necessary to get global asymptotic stability of the estimation error.

Error dynamics provides intuitive evolution

- Error dynamics depends on green parameters

$$\begin{aligned} \dot{e} &= \ell A e + \Delta_\varphi(x, e, u) - \ell K \,\mathrm{sat}_{\bar{\sigma}}(Ce) \\ \dot{\sigma} &= -\ell\lambda\sigma + \ell(\lambda + \epsilon) e^T C^T C e, \qquad \sigma \in \mathbb{R}_{\geq 0} \end{aligned}$$

- Parameters ℓ, K are the standard parameters of the High-Gain Observer
- Parameter $\lambda > 0$ forces the saturation level σ to converge exponentially to zero if $e_y = Ce = 0$
- Parameter $\lambda + \epsilon > 0$ causes necessary increase of σ wen $e_y \neq 0$ so that the error dynamics can be stabilized.
- Observer design involves the selection of $\ell, K, \lambda, \epsilon$.
- **Redesign Philosophy:** fix ℓ, K and then find suitable λ, ϵ.

GES of error dynamics uses sector conditions

- We make a time-rescaling $t \to \tau := \ell t$. Now $\dot{e} \to e'$
- Equivalent expression of error dynamics:

$$\begin{aligned} e' &= (A - KC)e + \ell^{-1}\Delta_\varphi(x, e, u) + Kq \\ \sigma' &= -\lambda\sigma + (\lambda + \epsilon)e^T C^T Ce, \quad \sigma \in \mathbb{R}_{\geq 0} \\ q &= Ce - \mathrm{sat}_{\bar{\sigma}}(Ce) \end{aligned}$$

- Two cases: $q = 0$ when $\bar{\sigma} \geq |Ce|$ and $q > 0$ when $\bar{\sigma} < |Ce|$
- Saturation inequality: $|q| \;\leq\; |e_y| \;\leq\; |Ce|$
- Stability is proven using a nonsmooth Lyapunov function

$$V(e, \sigma) = \underbrace{e^T Pe + \zeta\sigma}_{V_{REG}(e,\sigma)} + \underbrace{\eta \max\{e^T C^T Ce - \sigma, 0\}}_{V_{GLOB}(e,\sigma)},$$

Main stability theorem

Theorem. Fix any K such that $A - KC$ is Hurwitz and any $\lambda > 0$. There exists a $\ell^\star \geq 1$ and a $\epsilon^\star$ such that for any $\ell > \ell^\star$, $\epsilon > \epsilon^\star$ the following holds with $\nu = 0$ (no measurement noise)

$$|\hat{x}_i(t) - x_i(t)| \;\leq\; \alpha_1 \boldsymbol{\ell^{i-1}} \exp(-\alpha_2 \boldsymbol{\ell}\, t)|\hat{x}(0) - x(0)| + + \frac{\alpha_3 |d|_\infty}{\boldsymbol{\ell^{n-(i-1)}}}$$

Proof Sketch: $V(e,\sigma) = \underbrace{e^T P e + \zeta\sigma}_{V_{REG}} + \underbrace{\eta \max\{e^T C^T C e - \sigma, 0\}}_{V_q}$

- Use V_{REG} when $q = 0$ (where $V_q = 0$) and fix $\ell^\star, \zeta$
- Use V_q when $q > 0$ and fix $\epsilon^\star, \eta$
- both steps prove $\dot{V}(e,\sigma) \leq -c_1 V + c_2 \ell^{-2n} |d|_\infty^2$

Remarks:

- The Proof method extends from the linear case
- ISS bounds are established here that carry over to the linear case

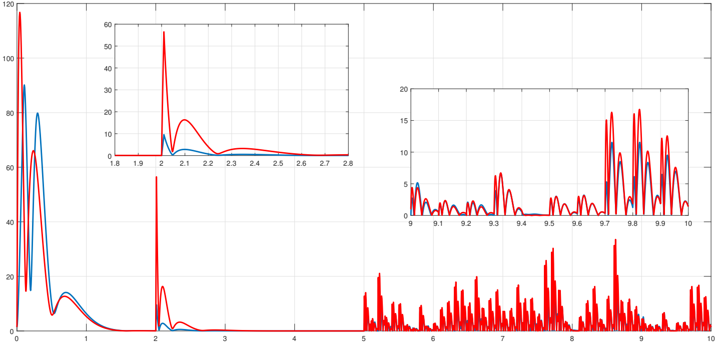
Noise analysis

What about when $y = x_1 + \nu$ where ν is some measurement noise? We can show that the following bound holds

$$|\hat{x}_i(t) - x_i(t)| \leq \alpha_1 \ell^{i-1} \exp(-\alpha_2 \ell\, t)|\hat{x}(0) - x(0)| + \frac{\alpha_3 |d|_\infty}{\ell^{n-(i-1)}} + \alpha_4 \ell^{i-1} |\nu|_\infty$$

- The Stubborn redesign preserves ISS properties w.r.t. measurement noise
- The stubborn redesign improves performances in presence of outliers (see forthcoming simulations)

Example: forced Duffing oscillator



- $t \in [0, 2)$: convergence with $d = 0$, $\nu = 0$
- $t = 2$: impulsive outliers $\nu \neq \delta(t)$
- $t \in [5, 10)$: high-frequency measurement noise
- **Red line:** standard high-gain observer ($|\hat{x}(t) - x(t)|$)
- **Blue line:** stubborn high-gain observer ($|\hat{x}(t) - x(t)|$)

Conclusions

- A new saturated output injection scheme with dynamic saturation level
- Proof involves a nonsmooth Lyapunov function and use of generalized sector conditions
- Simulations confirm usefulness with outliers/spikes in measured signals.
- Simulations does not show any deterioration of the performances in presence of white/coloured measurement noise
- Main limitations and future work:
  - how to quantify performance wrt outliers?
  - extension to other classes of nonlinear observers