Stability and stabilization of Poisson jump linear systems

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Summary

- Linear systems with Poisson jumps (PJLS)
- Mean square stability
- Numerical example
- State-feedback mean square stabilization
- Numerical example
- Conclusions

Stability and stabilization of Poisson jump linear systems

Consider the PJLS:

$$\dot{\mathbf{x}}(t) = A_{\theta(t)}\mathbf{x}(t) + B_{\theta(t)}\mathbf{u}(t)$$

- $\mathbf{x}(t) \in \mathbb{R}^n$ is the state
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the control input variable
- $\theta(t) \in \mathscr{N} = \{1, 2, \ldots, N\}$ is a switching signal

Assumptions on the switching signal

The switching signal $\theta(t)$ is a stochastic process subject to mode-dependent Poisson jumps.

Letting $t_k, k \in \{0, 1, 2, \ldots\}$, $t_0 = 0$, be the sequence of jump times, the dwell-time $\tau_k = t_{k+1} - t_k$ is exponentially distributed with parameter $\lambda_{\theta(t_k)} > 0$.

$$\Pr\{\tau_k \geq \Delta\} = \exp(-\lambda_{\theta(t_k)}\Delta)$$

Note that $E[\tau_k] = \lambda_{\theta(t_k)}^{-1}$.

Comparison with Markov Jumps Linear Systems

In a MJLS, the switching signal $\theta(t)$ is a Markov chain with transition rate matrix Λ.
Both the distribution of the dwell-time τ_k (exponential) and the transition probability between modes are specified.

In a PJLS only the distribution of the dwell-time τ_k is specified (exponential).

No information is assumed on the transition probabilities between modes.

PJLS as an uncertain positive MJLS

In a PJLS, $\theta(t)$ can be seen as an inhomogeneous Markov process with uncertain transition rate matrix:

$$\Lambda(t) = \begin{bmatrix} -\lambda_1 & ? & \cdots & ? \\ ? & -\lambda_2 & \cdots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \cdots & -\lambda_N \end{bmatrix}$$

A PJLS is an uncertain MJLS with $\Lambda(t) \in \mathrm{co}\{\bar{\Lambda}_k\}$.

The vertices $\bar{\Lambda}_k$ are obtained from all combinations of rows having a single nonzero off-diagonal entry λ_i.

The number of vertices $\bar{\Lambda}_k$ is $\bar{N} = (N-1)^N$.

Example of matrix vertices - $N = 3$

$$\bar{\Lambda}_1 = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ \lambda_2 & -\lambda_2 & 0 \\ \lambda_3 & 0 & -\lambda_3 \end{bmatrix}, \bar{\Lambda}_2 = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ \lambda_2 & -\lambda_2 & 0 \\ 0 & \lambda_3 & -\lambda_3 \end{bmatrix}$$

$$\bar{\Lambda}_3 = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_3 \end{bmatrix}, \bar{\Lambda}_4 = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ 0 & \lambda_3 & -\lambda_3 \end{bmatrix}$$

$$\bar{\Lambda}_5 = \begin{bmatrix} -\lambda_1 & 0 & \lambda_1 \\ \lambda_2 & -\lambda_2 & 0 \\ \lambda_3 & 0 & -\lambda_3 \end{bmatrix}, \bar{\Lambda}_6 = \begin{bmatrix} -\lambda_1 & 0 & \lambda_1 \\ \lambda_2 & -\lambda_2 & 0 \\ 0 & \lambda_3 & -\lambda_3 \end{bmatrix}$$

$$\bar{\Lambda}_7 = \begin{bmatrix} -\lambda_1 & 0 & \lambda_1 \\ 0 & -\lambda_2 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_3 \end{bmatrix}, \bar{\Lambda}_8 = \begin{bmatrix} -\lambda_1 & 0 & \lambda_1 \\ 0 & -\lambda_2 & \lambda_2 \\ 0 & \lambda_3 & -\lambda_3 \end{bmatrix}$$

Addressed problems

- Mean square stability of a PJLS
- State-feedback mean square stabilization of a PJLS

Mean square stability (MS-stability)

Definition

The PJLS

$$\dot{\mathbf{x}}(t) = A_{\theta(t)}\mathbf{x}(t)$$

is MS-stable if

$$\lim_{t\to\infty} E[\mathbf{x}(t)^\top \mathbf{x}(t)] = 0$$

for any $\mathbf{x}(0)$ and all admissible $\theta(t)$.

MS-stability is equivalent to second-moment stability, i.e. asymptotic convergence to zero of any squared norm of $\mathbf{x}(t)$.

Necessary and sufficient condition for M-stability

Theorem 1
MS-stability of the PJLS is equivalent to stability under arbitrary switching of the deterministic system of order Nn^2:

$$\dot{\xi}(t) = \Phi_{\sigma(t)}\xi(t)$$

with

$$\begin{aligned} \Phi_k &= \mathrm{diag}\{A_i \oplus A_i\} + \bar{\Lambda}_k^\top \otimes I_{n^2} \\ \sigma(t) \in \Sigma &= \{1, 2, \ldots, (N-1)^N\} \end{aligned}$$

Proof of Theorem 1

Let $S_i(t) = E[x(t)x(t)^\top \mathscr{I}_{\sigma(t)=i}]$. Then

$$\dot{S}_i(t) = A_i S_i + S_i A_i^\top + \sum_{j=1}^{N} \lambda_{ji}(t) S_j$$

Let $\xi(t) = Vec\{S_i\}$, $\bar{N} = (N-1)^N$, $\Lambda(t) = \sum_{k=1}^{\bar{N}} \bar{\Lambda}_k \alpha_k(t)$.

$$\dot{\xi}(t) = \left(\sum_{k=1}^{\bar{N}} \alpha_k(t) \Phi_k \right) \xi(t), \quad \Phi_k = \mathrm{diag}\{A_i \oplus A_i\} + \bar{\Lambda}_k^\top \otimes I_{n^2}$$

Stability of this polytopic system is equivalent to stability under arbitrary switching of the switching system

$$\dot{\xi}(t) = \Phi_\sigma \xi(t), \quad \sigma = \{1, 2, \cdots, \bar{N}\}$$

Sufficient conditions for MS-stability

Theorem 2

The PJLS is MS-stable if one of the following conditions is satisfied:

- (i) There exist $\mathbf{Q} \succ 0$ such that

$$\Phi_k \mathbf{Q} + \mathbf{Q} \Phi_k^\top \prec 0, \quad k \in \Sigma$$

- (ii) There exist strictly positive definite matrices $\mathbf{P}_i$, $i \in \mathscr{N}$, such that

$$A_i^\top \mathbf{P}_i + \mathbf{P}_i A_i + \lambda_i (\mathbf{P}_j - \mathbf{P}_i) \prec 0, \quad i, j \in \mathscr{N}, j \neq i$$

- (iii) There exist strictly positive definite matrices $\mathbf{S}_i \in \mathbb{R}_+^n$, $i \in \mathscr{N}$, such that

$$A_i \mathbf{R}_i + \mathbf{R}_i A_i^\top - \lambda_i \mathbf{R}_i + \sum_{j=1, j \neq i}^{N} \lambda_j \mathbf{R}_j \prec 0, \quad i \in \mathscr{N}$$

Remarks on Theorem 2

- The conditions (i) of Theorem 2 imply the existence of a common quadratic Lyapunov function $V(\xi) = \xi'\mathbf{Q}^{-1}\xi$, for the deterministic system.
- The conditions (ii) of Theorem 2 imply the existence of the stochastic Lyapunov function $V(x,\theta) = x'\mathbf{P}_\theta x$ for the stochastic system, or, equivalently the K- co-positive common linear Lyapunov function $V(\xi) = \mathbf{p}^\top \xi$, $\mathbf{p} = \mathrm{colvec}\{\mathbf{P}_i\}$ for the deterministic system. Since $\mathbf{p}'\Phi_k \overset{\mathscr{K}}{\ll} 0$, it results $\dot{V}(\xi) < 0$.
- The conditions (iii) of Theorem 2 implies $\Phi_k \mathbf{r} \overset{\mathscr{K}}{\ll} 0$, where $\mathbf{r} = \mathrm{colvec}\{\mathbf{R}_i\}$ and hence stability under arbitrary switching of system $\dot{\xi} = \Phi_{\sigma(t))}\xi$.

Relations via cone-invariance theory

$$\mathbf{r} = \mathrm{colvec}\{\mathbf{R}_i\}, \quad \mathbf{p} = \mathrm{colvec}\{\mathbf{P}_i\}$$

$$\Psi\mathbf{r} \overset{\mathcal{K}}{\ll} 0 \longrightarrow \Phi_k \mathbf{r} \overset{\mathcal{K}}{\ll} 0$$

$$\Psi\mathbf{r} \overset{\mathcal{K}}{\ll} 0 \longleftrightarrow \mathbf{p}^\top \Psi \overset{\mathcal{K}}{\ll} 0 \longrightarrow \mathbf{p}^\top \Phi_k \overset{\mathcal{K}}{\ll} 0$$

$$\Psi\mathbf{r} \overset{\mathcal{K}}{\ll} 0 \longleftrightarrow \Phi_k Q + Q\Phi_k^\top \prec 0$$

where $\Psi \overset{\mathcal{K}}{>} \Phi_k$ and

$$Q = \mathrm{diag}\{\bar{Q}_i \otimes \bar{Q}_i\}, \quad \bar{Q}_i = P_i^{-1/2}\left(P_i^{1/2} R_i P_i^{1/2}\right)^{1/2} P_i^{-1/2}$$

Numerical example - MS-stability

Consider the PJLS with $N = 3$, and

$$A_1 = \begin{bmatrix} -1 & 0 \\ -1 & 0.1 \end{bmatrix}, A_2 = \begin{bmatrix} 0.1 & -1 \\ 0 & -2 \end{bmatrix}, A_3 = \begin{bmatrix} -2 & 0.2 \\ -1 & -2 \end{bmatrix}$$

and $\quad \lambda_1 = \lambda_2 = 1, \quad \lambda_3 = \text{free}.$

MS-stability is guaranteed for:

- Theorem 2(i): $\lambda_3 \leq 12.33$
- Theorem 2(ii): $\lambda_3 \leq 12.43$
- Theorem 2(iii): $\lambda_3 \leq 0.16$

Necessary condition (stability of the vertices $\Phi_k, k \in \Sigma$):
$\lambda_3 \leq 12.46$

Example

20 realizations of the squared 2-norm of the state with $x(0) = [0.5\ 0.5]'$ and $\lambda_3 = 10$ [$\lambda_3 = 20$], illustrating MS-stability [instability]. The switching signal $\theta(t)$ commutes between modes 2 and 3, that corresponds to the worst sub-MJLS composed of a pair of modes.

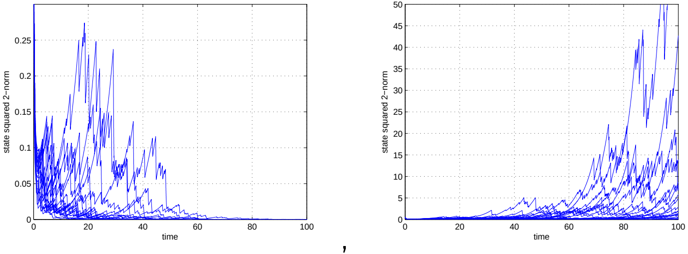

Figure 1 : 20 realizations of the state squared 2-norm of the PJLS of Example 1 with $\lambda_3 = 10$ (left) and $\lambda_3 = 20$ (right).

Scalar case

Remark 1: In the scalar case, stability of all modes ($A_i = a_i < 0, \forall i$) implies MS-stability of the PJLS. Moreover, the presence of two unstable (or marginally stable) modes prevents MS stability. As a matter of fact, the sub-MJLS composed by these two modes cannot be MS-stable. Then, without loss of generality, we assume that there is only one nonnegative a_i, say $a_u \geq 0$. It can be proven that the PJLS is MS-stable if and only if

$$\lambda_u > a_u \max_{i \neq u}(2 - \frac{\lambda_i}{a_i})$$

The interpretation of this condition is that the average dwell-time in the unstable mode must be sufficiently small.

State-feedback MS-stabilization

Definition

The PJLS

$$\dot{\mathbf{x}}(t) = A_{\theta(t)}\mathbf{x}(t) + B_{\theta(t)}\mathbf{u}(t)$$

is MS-stabilizable if there exists a feedback law

$$\mathbf{u}(t) = K_{\theta(t)}\mathbf{x}(t)$$

such that the closed-loop system is MS-stable for all admissible $\theta(t)$.

Sufficient conditions for MS-stabilization

Theorem 3

If there exist strictly positive definite matrices $\mathbf{R}_i$, and $\mathbf{W}_i$, $i \in \mathcal{N}$, such that

$$A_i\mathbf{R}_i + \mathbf{R}_i A_i - \lambda_i \mathbf{R}_i + B_i \mathbf{W}_i + \mathbf{W}_i' B_i' + \sum_{j=1, j\neq i}^{N} \lambda_j \mathbf{R}_j \prec 0, \quad i \in \mathcal{N}$$

for all $i, j \in \mathcal{N}$ with $j \neq i$, then the gain matrices K_i, $i \in \mathcal{N}$, defined by

$$K_i = \mathbf{W}_i \mathbf{R}_i^{-1}$$

make the closed-loop system positive and MS-stable.

Sufficient conditions for MS-stabilization

Theorem 4

If there exist a positive definite matrix $Q = \mathrm{diag}\{\mathbf{Q}_i\}$, a matrix $\mathbf{W} = \mathrm{diag}\{\mathbf{W}_i\}$ such that, $\forall k \in \Sigma$,

$$\Phi_k \mathbf{Q} + \mathbf{Q}\Phi_k^\top + \tilde{B}\mathbf{W} + \mathbf{W}^\top \tilde{B}^\top \quad \prec \quad 0$$

then, the gain matrices $K_i = \mathbf{W}_i \mathbf{Q}_i^{-1}$, $i \in \mathcal{N}$ make the closed-loop system positive and M-stable.

$$\tilde{B} = \mathrm{diag}\{\tilde{B}_i\}, \quad \tilde{B}_i = (B_i \otimes I) + (I \times B_i)J, \quad J:\ (x \otimes u) = J(u \otimes x)$$

Conclusions

- The class of linear systems subject to Poisson jumps (PJLS) has been studied.
- It has been shown that MS-stability of a PJLS is equivalent to stability under arbitrary switching of a higher order deterministic system.
- Sufficient conditions for MS-stability and MS-stabilization have been derived.
- Possible extensions concern performance optimization of input/output norms.

Thanks for your attention!