Harmonic influence in social networks
Identification of influencers by message passing

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based on joint works with
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What is social influence?

What is the most influential node in a network?

Context-dependent question:
opinion dynamics // epidemic spread // cascading activation // resource competition // ...

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**In this talk:**
A leader competes against an adversary field to influence the opinions of the other individuals

Our approach: harmonic influence

Which leader location (node) maximizes the influence?

1. We define the harmonic influence of a node
2. We relate social and electrical networks
3. We derive a message-passing algorithm
4. We prove its convergence
5. We discuss a few simulations


Influence Maximization
Opinions in the social network

Each individual $i$ has **opinion** $x_i(t) \in \mathbb{R}$ evolving with time

Opinions evolve through

- **social interactions** between individuals
- influence of an external field

Weighted graph $G = (I, E, C)$
- node set $I = \{f, 1, 2, \ldots, n\}$
- $f$ is a special **field** node
- undirected edge set $E$
- **non-negative weight matrix** $C$
  such that $C_{ij}C_{ji} > 0 \iff \{i, j\} \in E$
Opinions dynamics in the social network

We introduce a leader against the field

\[ x_f(t) = x_f(0) \quad \text{for all} \quad t \]

The leader \( \ell \) is also stubborn

\[ x_\ell(t) = x_\ell(0) \quad \text{for all} \quad t \]

The remaining individuals do local averaging

\[ x_i(t+1) = \sum_{j \neq i} Q_{ij} x_j(t) \]

where \( Q = D - I \)

\[ D = \text{diag}(C) \]
Opinions dynamics in the social network

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- The leader $\ell$ is also stubborn
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- The remaining individuals do local averaging
  \[ x_i(t+1) = \sum_{j \neq i} Q_{ij} x_j(t) \]

where $Q = D^{-1}C$

with diagonal matrix $D = \text{diag}(C\mathbf{1})$
Harmonic influence

Let Laplacian matrix \( L = D - C \)
Normalize opinions in \([0, 1]\)

**Dirichlet problem**

Equilibrium opinions solve Laplacian system with boundary conditions

\[
\begin{align*}
L \mathbf{x} &= \mathbf{0} \\
x_{\ell} &= 1 \\
x_f &= 0
\end{align*}
\]

The *Harmonic Influence of* \( \ell \) is \( H(\ell) := \mathbf{1}^T \mathbf{x} \)

(\( \mathbf{x} \) is said to be a harmonic function)

Computing \( H \) requires solving \( n \) linear systems, one for each possible leader
Computing the Harmonic Influence

Problem:
Find an algorithm that
- solves all $n$ systems at the same time
- is distributed
Computing the Harmonic Influence

Problem:
Find an algorithm that

- solves all $n$ systems at the same time
- is distributed

Solution:
Message-passing iterative algorithm that approximates $H$

- with provable convergence
- with insights on convergence speed and approximation error
Electrically-inspired Message-Passing Algorithm
Electrical analogy (assuming $C^\top = C$)

Equilibrium opinions $\mathbf{x}$ are the potentials of an electrical network.

$\ell$

$\mathbf{f}$
Electrical analogy (assuming $C^\top = C$)

Equilibrium opinions $x$ are the potentials of an electrical network

- node $f$ has potential 0
- node $\ell$ has potential 1
- conductances of value $C_{ij} = C_{ji}$ substitute each edge

Computation of $H(\ell)$ on trees:
1. compute the effective resistances
2. compute the current leaving $\ell$
3. compute all potentials
4. sum up potentials to get $H(\ell)$
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**Computation of $H(\ell)$ on trees:**

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Propagating of potentials: from leaves to root

Also $H(\ell)$ can be computed recursively, from the leaves to the root.
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**Notation:**

- $H_{i \rightarrow j}$: $H(i)$ on the graph without edge $\{i, j\}$
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**Notation:**

- $H^{i \rightarrow j}$: $H(i)$ on the graph without edge $\{i, j\}$
- $W^{i \rightarrow j}$: potential of $i$ if $j$ is at potential 1
Propagation of potentials: from leaves to root

Also $H(\ell)$ can be computed recursively, from the leaves to the root

**Notation:**

- $H^{i\rightarrow j}$: $H(i)$ on the graph without edge $\{i, j\}$
- $W^{i\rightarrow j}$: potential of $i$ if $j$ is at potential 1
For simplicity, $C_{ij} = 1$ for all $\{i, j\} \in E$
Propagation of potentials: example

For simplicity, \( C_{ij} = 1 \) for all \( \{i, j\} \in E \)

\[
\begin{align*}
H_{k \rightarrow i} &= 1 \\
W_{k \rightarrow i} &= 1
\end{align*}
\]
Propagation of potentials: example

For simplicity, $C_{ij} = 1$ for all $\{i, j\} \in E$

$H^{k\rightarrow i} = 1 \quad W^{k\rightarrow i} = 1$
$H^{f\rightarrow i} = 0 \quad W^{f\rightarrow i} = 0$
Propagation of potentials: example

For simplicity, $C_{ij} = 1$ for all $\{i, j\} \in E$

$$
H_{k \to i} = 1 \quad W_{k \to i} = 1 \\
H_{f \to i} = 0 \quad W_{f \to i} = 0 \\
H_{i \to j} = 1 + W_{k \to i} H_{k \to i} + W_{f \to i} H_{f \to i}
$$
For simplicity, $C_{ij} = 1$ for all $\{i, j\} \in E$

$H^{k \rightarrow i} = 1 \quad W^{k \rightarrow i} = 1$

$H^{f \rightarrow i} = 0 \quad W^{f \rightarrow i} = 0$

$H^{i \rightarrow j} = 1 + W^{k \rightarrow i} H^{k \rightarrow i} + W^{f \rightarrow i} H^{f \rightarrow i}$

$W^{i \rightarrow j} = \frac{1}{1 + (1 - W^{k \rightarrow i}) + (1 - W^{f \rightarrow i})}$
Propagation of potentials: example

For simplicity, \( C_{ij} = 1 \) for all \( \{i, j\} \in E \)

\[
\begin{align*}
H^{k \rightarrow i} &= 1 \quad W^{k \rightarrow i} = 1 \\
H^{f \rightarrow i} &= 0 \quad W^{f \rightarrow i} = 0 \\
H^{i \rightarrow j} &= 1 + W^{k \rightarrow i} H^{k \rightarrow i} + W^{f \rightarrow i} H^{f \rightarrow i} \\
W^{i \rightarrow j} &= \frac{1}{1 + (1 - W^{k \rightarrow i}) + (1 - W^{f \rightarrow i})} \\
H^{j \rightarrow \ell} &= 1 + W^{i \rightarrow j} H^{i \rightarrow j} + W^{i' \rightarrow j} H^{i' \rightarrow j} \\
&= 1 + W^{i \rightarrow j} + W^{i' \rightarrow j} + W^{k \rightarrow j} + W^{f \rightarrow j} + W^{h \rightarrow j} + W^{h' \rightarrow j} \\
\text{because} \quad W^{k \rightarrow j} &= W^{k \rightarrow i} W^{i \rightarrow j}
\end{align*}
\]
For simplicity, $C_{ij} = 1$ for all $\{i, j\} \in E$

\[
H_{k \rightarrow i} = 1 \quad W_{k \rightarrow i} = 1
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H^{j \rightarrow \ell} &= 1 + W^{i \rightarrow j} H^{i \rightarrow j} + W^{i' \rightarrow j} H^{i' \rightarrow j} \\
W^{j \rightarrow \ell} &= \frac{1}{1 + (1 - W^{i \rightarrow j}) + (1 - W^{i' \rightarrow j})} \\
H(\ell) &= 1 + W^{j \rightarrow \ell} H^{j \rightarrow \ell}
\end{align*}
\]
Message Passing Algorithm

Generic graph $\mathcal{G} = (I, E, C)$  \hspace{1cm} $C$ needs not be symmetric

Node $i$ sends to neighbor $j$ two messages:
- $W^{i \rightarrow j}(t)$: estimate of $x_i$ if $\ell = j$
- $H^{i \rightarrow j}(t)$: estimate of $H(i)$ in the graph $\mathcal{G} \setminus \{i,j\}$

Message-Passing Algorithm

<table>
<thead>
<tr>
<th>boundary</th>
<th>$W^{i \rightarrow j}(t) = 0$, $H^{i \rightarrow j}(t) = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>initialization</td>
<td>$W^{i \rightarrow j}(0) = 1$, $H^{i \rightarrow j}(0) = 1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>update</th>
<th>$W^{i \rightarrow j}(t + 1) = \left[1 + \sum_{k \in N_i \setminus j} \frac{C_{ik}}{C_{ij}} (1 - W^{k \rightarrow i}(t))\right]^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H^{i \rightarrow j}(t + 1) = 1 + \sum_{k \in N_i \setminus j} W^{k \rightarrow i}(t) H^{k \rightarrow i}(t)$</td>
</tr>
</tbody>
</table>

| estimate  | $H_t(\ell) = 1 + \sum_{i \in N_\ell} W^{i \rightarrow \ell}(t) H^{i \rightarrow \ell}(t)$ |
Analysis of the MPA
Theorem

Let $G = (I, E, C)$ be any connected graph with symmetric $C$. Then, the Message Passing Algorithm converges.

Proof outline:

1. define an MPA-like dynamics on directed graphs $M$
2. define suitable message digraph $M_G$, that describes the topology of the dependences between messages
3. prove the convergence of the MPA-like dynamics induced on $M_G$:
   - when acyclic (by construction)
   - when strongly connected (more difficult)
   - in general (combining the sub-proofs)
Proof 1/3: MPA-like dynamics

- Digraph $\mathcal{M} = (V, \Phi)$, its adjacency matrix $M \in \{0, 1\}^{V \times V}$
- Vectors $\mathbf{r}, \mathbf{s} \in \mathbb{R}^V_{>0}$, such that $r_v = s_v^{-1}$, and

$$W = \text{diag}(\mathbf{r})M\text{diag}(\mathbf{s})$$

- Two sequences of non-negative vectors $\alpha(t), \beta(t)$, such that $\alpha(t)$ is non-decreasing in every component and $\beta(t)$ is convergent.

MPA-like is $\omega(t) \in (0, 1]^V$ and $\eta(t) \in [1, +\infty)^V$ such that

$$\omega(0) = \eta(0) = 1$$

$$\omega_v(t + 1) = \frac{1}{1 + \alpha_v(t) + \sum_w W_{vw} (1 - \omega_w(t))}$$

$$\eta_v(t + 1) = 1 + \beta_v(t) + \sum_w M_{vw} \omega_w(t) \eta_w(t)$$
Proof 2/3: Message digraph $\mathcal{M}_G$

$\mathcal{G}$

Social graph $\mathcal{G} = (I, E)$

$\mathcal{M}_G$

Message digraph $\mathcal{M}_G = (\tilde{E}, \Phi)$

$\tilde{E} = \{ji : \{i, j\} \in E, i \neq f\}$

$\Phi = \{(ji, ik) : ji, ik \in \tilde{E}, j \neq k\}$
Proof 2/3: Message digraph $\mathcal{M}_G$

Social graph $\mathcal{G} = (I, E)$

Message digraph $\mathcal{M}_G = (\vec{E}, \Phi)$

The messages $W_{i\to j}(t)$ and $H_{i\to j}(t)$ are associated to node $ji$ in $\mathcal{M}_G$.

The counterpart of the constant message $W_{f\to k}(t) = 0$ is the (constant) sequence $\alpha_{ik} = C_{kf}/C_{ki} > 0$.
Proof 3/3: analysis on any digraph $\mathcal{M}$

If $\mathcal{M}$ acyclic $\iff$ convergence

(follow partial order)
If $\mathcal{M}$ acyclic $\implies$ convergence

If $\mathcal{M}$ is strongly connected and contains $kh$ where $\alpha_{kh}(t) > 0$ $\implies$ convergence

(W-messages have limits by monotonicity; update matrix for H-messages non-negative irreducible and eventually Shur stable)
Proof 3/3: analysis on any digraph $\mathcal{M}$

If $\mathcal{M}$ acyclic $\implies$ convergence

If $\mathcal{M}$ is strongly connected and contains $kh$ where $\alpha_{kh}(t) > 0$ $\implies$ convergence

If every node in a non-trivial strongly connected component of $\mathcal{M}$ can reach $kh$ where $\alpha_{kh}(t) > 0$ $\implies$ convergence

(Condense components, use partial order, compose previous results)
Proof 3/3: analysis on any digraph $\mathcal{M}$

If $\mathcal{M}$ acyclic $\implies$ convergence

If $\mathcal{M}$ is strongly connected and contains $kh$ where $\alpha_{kh}(t) > 0$ $\implies$ convergence

If every node in a non-trivial strongly connected component of $\mathcal{M}$ can reach $kh$ where $\alpha_{kh}(t) > 0$ $\implies$ convergence

$\mathcal{M}_G$ satisfies these assumptions $\implies$ the MPA converges
Simulations
Simulations: random tree

Random tree graph: 50 nodes, 49 edges, diameter=13, $C_{ij} = 0.05$ for all $i$

Convergence time = diameter
Simulations: random tree

Random tree graph: 50 nodes, 49 edges, diameter=13, $C_{ij} = 0.05$

**Left:** true $H(\ell)$ vs. estimate $H_\infty(\ell)$

**Right:** true potential $W^{i\to\ell}$ vs. $W_\infty^{i\to\ell}$

MPA is *exact* on trees
Random addition of 10 edges: 50 nodes, 59 edges, $C_{if} = 0.05$

Convergence time of $W^{i\to j}(t)$ increases slightly
Convergence time of $H^{i\to j}(t)$ increases significantly
Simulations: graph with few cycles

Random addition of 10 cycles: 50 nodes, 59 edges, $C_{ij} = 0.05$

**Left:** true $H(\ell)$ vs. estimate $H_\infty(\ell)$

**Right:** true potential $W^i\rightarrow\ell$ vs. $W^i_\infty\rightarrow\ell$
Simulations: Erdős-Rényi random graph

Erdős-Rényi random graph: 50 nodes, 131 edges, $C_{ij} = 0.05$

Convergence time of $W^{i \rightarrow j}(t)$ almost unchanges
Convergence time of $H^{i \rightarrow j}(t)$ increases significantly
Simulations: Erdős-Rényi random graph

Erdős-Rényi random graph: 50 nodes, 131 edges, $C_{if} = 0.05$

Left: true $H(\ell)$ vs. estimate $H_\infty(\ell)$

Right: true potential $W^{i\to\ell}$ vs. $W_\infty^{i\to\ell}$
Conclusions
Message-passing algorithm with two messages $H, W$
- designed on trees by an electrical analogy
- can be used on any undirected weighted graph $(I, E, C)$
- proved to converge if $C^\top = C$
- convergence in two phases: first messages $W$, then $H$
  - cycles degrade convergence speed (of $H$)
- cycles degrade (not too much) the accuracy of the approximation

More insights in:
W.S. Rossi and P. Frasca. Mean-field analysis of the convergence time of message-passing computation of harmonic influence in social networks, IFACWC, Toulouse, 2017
Refine analysis of MPA

- Extend convergence proof to non-symmetric networks
- Evaluate convergence time
- Estimate the error between convergence value and actual $H$

Improve design of MPA

- Accelerate convergence of $H^{i\rightarrow j}$ messages

*Can similar ideas be used for other centrality measures?*