The Kalman Decomposition for Linear Quantum Systems

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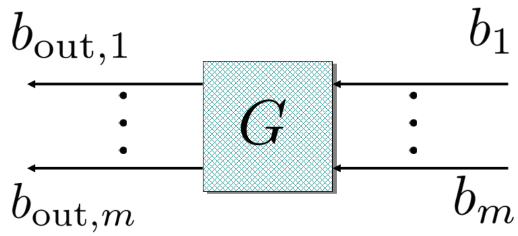
Introduction

- This paper extends the classical Kalman decomposition to the case of linear quantum systems.
- This involves introducing a coordinate transformation (change of variables) such that the transformed system is decomposed into four subsystems which represent different controllability and observability properties of the system.
- The transformed system is said to be in Kalman canonical form.
- Contrary to the classical case, the coordinate transformation used for the decomposition must belong to a specific class of transformations as a consequence of the laws of quantum mechanics.
- We propose a construction method for such transformations that put the system in a Kalman canonical form.
- Furthermore, we uncover an interesting structure for the obtained decomposition.

- The quantum Kalman decomposition naturally exposes decoherence-free modes, quantum-nondemolition modes, quantum-mechanics-free subspaces, and back-action evasion measurements in the quantum system, which are useful resources for quantum information processing, and quantum measurements.
- The theory developed is applied to physical examples.

Linear quantum systems



- A linear quantum system, is a collection of n quantum harmonic oscillators driven by m input boson fields.

- The mode of oscillator j, $j = 1, \ldots, n$, is described in terms of its annihilation operator $\boldsymbol{a}_j$, and its creation operator $\boldsymbol{a}_j^*$, the adjoint operator of $\boldsymbol{a}_j$. These are operators on an infinite-dimensional Hilbert space.

- These operators satisfy the *canonical commutation relations*

$$[\boldsymbol{a}_j(t),\ \boldsymbol{a}_k(t)] = 0, [\boldsymbol{a}_j^*(t),\ \boldsymbol{a}_k^*(t)] = 0, [\boldsymbol{a}_j(t),\ \boldsymbol{a}_k^*(t)] = \delta_{jk}.$$

- The system Hamiltonian $\boldsymbol{H}$ is given by

$$\boldsymbol{H} = (1/2)\breve{\boldsymbol{a}}^{\dagger}\Omega\breve{\boldsymbol{a}},$$

where $\breve{\boldsymbol{a}} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{a}^{\#} \end{bmatrix}, \breve{\boldsymbol{a}}^{\dagger} = [\boldsymbol{a}^{\dagger} \; \boldsymbol{a}^{\top}], \boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}, \boldsymbol{a}^{\#} = \begin{bmatrix} \boldsymbol{a}_1^* \\ \vdots \\ \boldsymbol{a}_n^* \end{bmatrix},$

$\boldsymbol{a}^{\dagger} = [\boldsymbol{a}_1^* \; \cdots \; \boldsymbol{a}_n^*]$ and Ω is a Hermitian matrix.

- The coupling of the system to the input fields is described by the vector of operatora $\boldsymbol{L} = [C_- \; C_+]\breve{\boldsymbol{a}}$, where C_- and C_+ are complex matrices.

- The input field are described in terms of annihilation operators $\boldsymbol{b}_k(t)$ and creation operators $\boldsymbol{b}_k^*(t)$ which also satisfy canonical commutation relations.

- The dynamics of the open linear quantum system is described by the following quantum stochastic differential equations (QSDEs)

$$\begin{aligned} \dot{\breve{\boldsymbol{a}}}(t) &= \mathscr{A}\breve{\boldsymbol{a}}(t) + \mathscr{B}\breve{\boldsymbol{b}}(t), \\ \breve{\boldsymbol{b}}_{\mathrm{out}}(t) &= \mathscr{C}\breve{\boldsymbol{a}}(t) + \mathscr{D}\breve{\boldsymbol{b}}(t), \end{aligned}$$

  where

$$\begin{aligned} \mathscr{D} &= I,\ \mathscr{C} = \begin{bmatrix} C_- & C_+ \\ C_+^\# & C_-^\# \end{bmatrix},\ \mathscr{B} = -\mathscr{C}^\flat,\ \text{and} \\ \mathscr{A} &= -\imath J_n \Omega - \frac{1}{2}\mathscr{C}^\flat \mathscr{C}. \end{aligned}$$

- Here

$$\mathscr{C}^\flat = J\mathscr{C}^\dagger J,\ J = \mathrm{diag}(I, -I),\ \mathscr{C}^\dagger = (\mathscr{C}^\#)^T.$$

- This is the complex annihilation-creation representation of the linear quantum system.

- There is another useful representation for the above quantum linear systems which is the real quadrature operator representation. It can be obtained from the annihilation-creation operator representation through the following transformations:

$$\begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix} \equiv \boldsymbol{x} \triangleq V\breve{\boldsymbol{a}}, \ \begin{bmatrix} \boldsymbol{q}_{\rm in} \\ \boldsymbol{p}_{\rm in} \end{bmatrix} \equiv \boldsymbol{u} \triangleq V\breve{\boldsymbol{b}}, \ \begin{bmatrix} \boldsymbol{q}_{\rm out} \\ \boldsymbol{p}_{\rm out} \end{bmatrix} \equiv \boldsymbol{y} \triangleq V\breve{\boldsymbol{b}}_{\rm out},$$

where $V \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -\imath I & \imath I \end{bmatrix}$.

- The operators $\boldsymbol{q}_i$ and $\boldsymbol{p}_i$ are called *conjugate* variables.

- They satisfy the canonical commutation relations

$$[\boldsymbol{q}_j(t), \boldsymbol{q}_k(t)] = 0, \ [\boldsymbol{p}_j(t), \boldsymbol{p}_k(t)] = 0, \ \text{and } [\boldsymbol{q}_j(t), \boldsymbol{p}_k(t)] = \imath\delta_{jk}.$$

- The QSDEs that describe the dynamics of the linear quantum system in the real quadrature operator representation are the following:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u}, \\ \boldsymbol{y} &= C\boldsymbol{x} + D\boldsymbol{u}, \end{aligned}$$

  where

$$\begin{aligned} D &= V\mathscr{D}V^\dagger = I,\ C = V\mathscr{C}V^\dagger,\ B = V\mathscr{B}V^\dagger = -C^\sharp, \\ A &= V\mathscr{A}V^\dagger = \mathbb{J}H - \frac{1}{2}C^\sharp C,\ X^\sharp \triangleq -\mathbb{J}X^\dagger\mathbb{J}, \mathbb{J} \triangleq \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]. \end{aligned}$$

- Here $H \triangleq V\Omega V^\dagger$ and the Hamiltonian can be written as $\boldsymbol{H} = (1/2)\boldsymbol{x}^\top H\boldsymbol{x}$.

- In the real quadrature representation the matrices A, B, C, D, and H are all real.
- The only coordinate transformations $\boldsymbol{x}_{\rm new} = S\boldsymbol{x}$ that preserve this structure are real symplectic; i.e.,
$$\mathbb{J}S^\dagger = S^\dagger \mathbb{J} S = \mathbb{J} \Leftrightarrow SS^\sharp = S^\sharp S = I.$$
- Also, since S is symplectic, it preserves the commutation relations.
- Also, it will be convenient to consider blockwise symplectic transformations; i.e.,
$$S^\top \mathbb{J} S = \begin{bmatrix} \mathbb{J} & 0 & 0 \\ 0 & \mathbb{J} & 0 \\ 0 & 0 & \mathbb{J} \end{bmatrix}. \qquad (1)$$

The Controllability and Observability Matrices

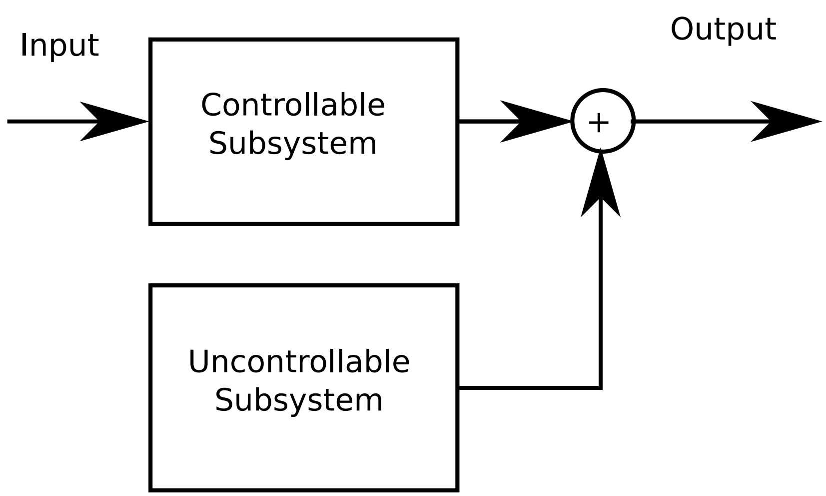
- The controllability and observability matrices for the linear quantum system are

$$C_G \triangleq \left[\mathscr{B} \ \mathscr{A}\mathscr{B} \ \cdots \ \mathscr{A}^{2n-1}\mathscr{B} \right],$$

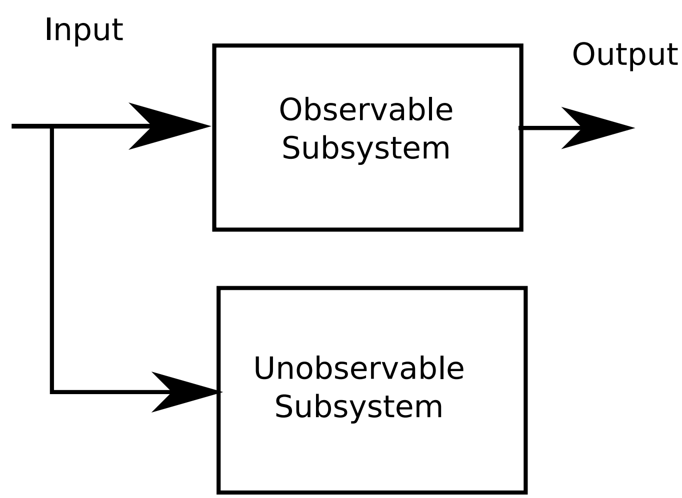
$$O_G \triangleq \begin{bmatrix} \mathscr{C} \\ \mathscr{C}\mathscr{A} \\ \vdots \\ \mathscr{C}\mathscr{A}^{2n-1} \end{bmatrix}.$$

- These matrices arise in classical linear systems theory and are used to determine if a given linear system has the property of **Controllability** or **Observability**.

- These notions play an important role in understanding the structure of classical linear systems and we show that they also play an important role in understanding the structure of quantum linear systems.



An uncontrollable linear system



An unobservable linear system

- The controllability and observability matrices also define corresponding subspaces which will be used in out quantum Kalman decomposition.
- $\mathrm{Im}(C_G)$ and $\mathrm{Ker}\,(O_G)$ are the controllable and unobservable subspaces of the space $\mathbb{C}^{2n}$.
- We define the uncontrollable and observable subspaces to be their *orthogonal complements* in $\mathbb{C}^{2n}$, that is $\mathrm{Ker}(C_G^\dagger)$, and $\mathrm{Im}(O_G^\dagger)$, respectively.

Quantum Notions related to the Controllability and Observability Subspaces

- We now introduce some important notions from quantum information science and quantum measurement theory which we will later show are naturally related to our Kalman decomposition of linear quantum systems.

Definition 1 *The linear span of the system variables related to the uncontrollable/unobservable subspace (the intersection of the uncontrollable and unobservable subspaces) of a linear quantum system is called its decoherence-free subsystem (DFS).*

- Decoherence-free subsystems for linear quantum systems have recently been studied by a number of authors.

- A natural extension of the notion of a QND variable is the following concept introduced by Tsang and Caves in 2012.

Definition 2 *The span of a set of observables* F_i, $i = 1, \ldots, r$, *is called a* quantum mechanics-free subsystem (QMFS) *if*

$$[F_i(t_1), F_j(t_2)] = 0$$

for all time instants $t_1, t_2 \in \mathbb{R}^+$, *and* $i, j = 1, \ldots, r$.

- We will show using our quantum Kalman decomposition that a QMFS corresponds to the linear span of the system variables related to the uncontrollable/observable subspace (the intersection of the uncontrollable and observable subspaces) of a linear quantum system.

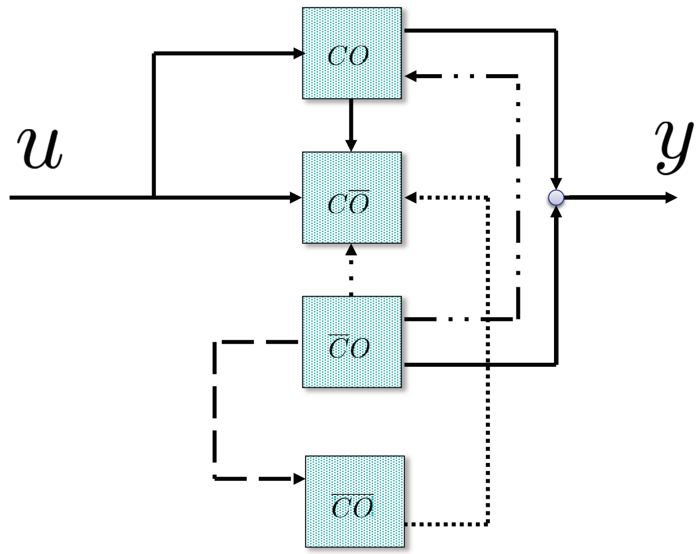
The Kalman decomposition in the real quadrature representation

Theorem 1 *There exists a real orthogonal and blockwise symplectic coordinate transformation* $\left[\begin{array}{c} \boldsymbol{q}_h \\ \boldsymbol{p}_h \\ \hline \boldsymbol{x}_{co} \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}} \end{array}\right] = S^\top \boldsymbol{x}$ *which after a re-arrangement transforms the linear quantum system into the form*

$$\begin{bmatrix} \dot{\boldsymbol{q}}_h(t) \\ \dot{\boldsymbol{x}}_{co}(t) \\ \dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) \\ \dot{\boldsymbol{p}}_h(t) \end{bmatrix} = \begin{bmatrix} A_h^{11} & A_{12} & A_{13} & A_h^{12} \\ 0 & A_{co} & 0 & A_{21} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{31} \\ 0 & 0 & 0 & A_h^{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_h(t) \\ \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{\bar{c}\bar{o}}(t) \\ \boldsymbol{p}_h(t) \end{bmatrix} + \begin{bmatrix} B_h \\ B_{co} \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u}(t),$$

$$\boldsymbol{y}(t) = [0 \; C_{co} \; 0 \; C_h] \begin{bmatrix} \boldsymbol{q}_h(t) \\ \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{\bar{c}\bar{o}}(t) \\ \boldsymbol{p}_h(t) \end{bmatrix} + \boldsymbol{u}(t).$$

- A block diagram of the corresponding Kalman decomposition is shown below.



- $\boldsymbol{q}_{h,i}, i = 1, \ldots, n_3$, are controllable but unobservable, while $\boldsymbol{p}_{h,i}, i = 1, \ldots, n_3$, are observable but uncontrollable.
- We see that every $c\bar{o}$ variable must have an associated $\bar{c}o$ variable. That is, they appear in conjugate pairs.
- Again the proof of the above theorem gives a constructive method for obtaining the required transformation matrix S.

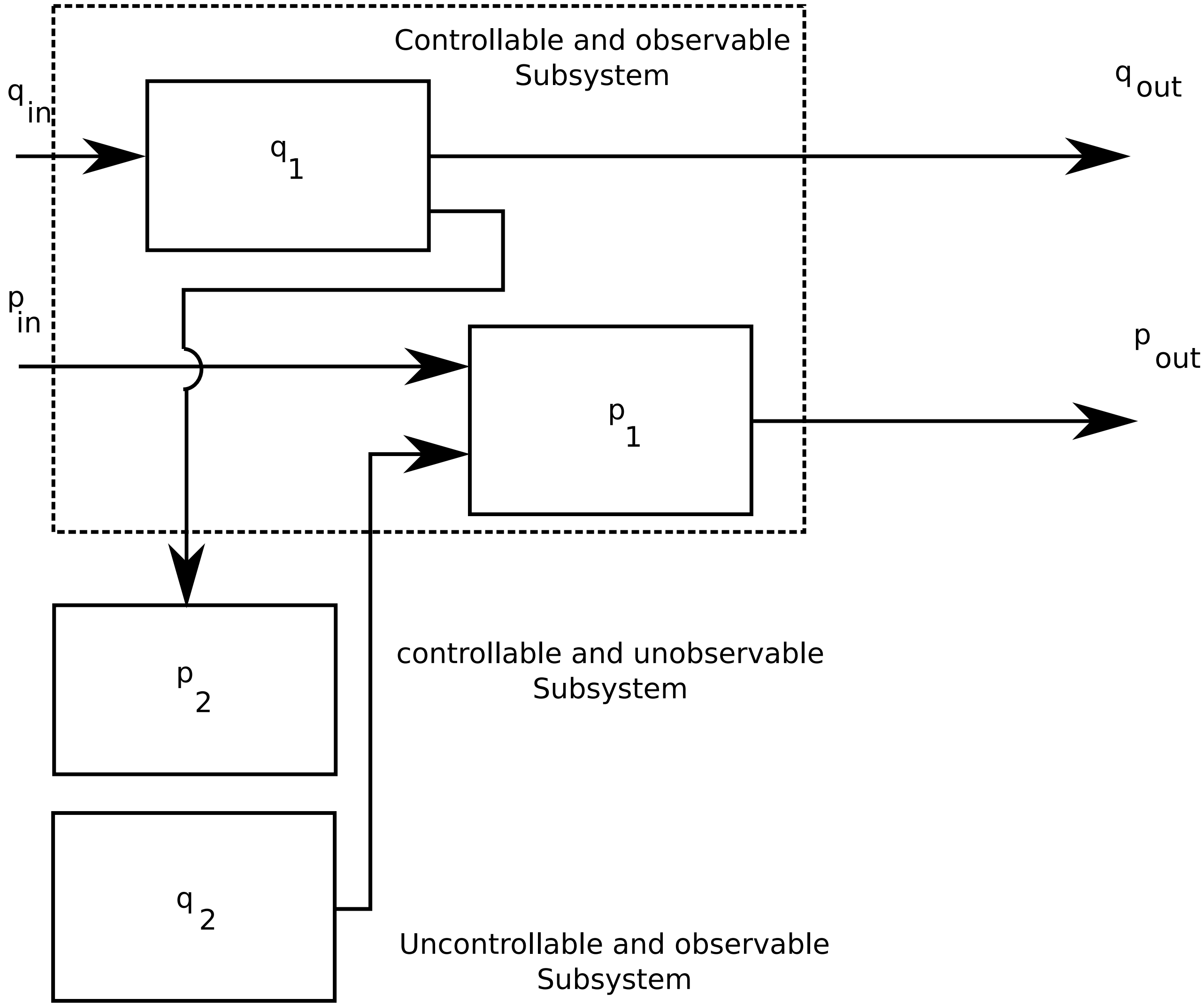
- Notice that the variables $\boldsymbol{p}_{h,i}$ evolve without any influence from the inputs or other system variables. Then, the set of $\boldsymbol{p}_{h,i}$, $i = 1, ..., n_3$, is a quantum mechanics free subsystem (QMFSS).
- This implies that each $\boldsymbol{p}_{h,i}$ is a QND variable.
- Moreover, the variables $\boldsymbol{x}_{\bar{c}\bar{o},i}$, $i = 1, \ldots, n_2$, are DF modes.

Example 1

- We now consider an using the real quadrature representation for which the Hamiltonian and coupling operator are given by $\boldsymbol{H} = 2\boldsymbol{q}_1\boldsymbol{q}_2$ and $\boldsymbol{L} = \frac{1}{\sqrt{2}}(\boldsymbol{q}_1 + \imath\boldsymbol{p}_1)$, respectively.
- We find that the system variables in the real quadrature representation form of the Kalman decomposition are given by $\boldsymbol{q}_h = -\boldsymbol{p}_2,\ \boldsymbol{p}_h = \boldsymbol{q}_2,\ \boldsymbol{q}_{co} = \boldsymbol{q}_1,\ \boldsymbol{p}_{co} = \boldsymbol{p}_1$.
- Also, the corresponding QSDEs are as follows:

$$
\begin{aligned}
\dot{\boldsymbol{p}}_2(t) &= -2\boldsymbol{q}_1(t), \\
\dot{\boldsymbol{q}}_1(t) &= -0.5\boldsymbol{q}_1(t) - \boldsymbol{q}_{\mathrm{in}}(t), \\
\dot{\boldsymbol{p}}_1(t) &= -0.5\boldsymbol{p}_1(t) - 2\boldsymbol{q}_2(t) - \boldsymbol{p}_{\mathrm{in}}(t), \\
\dot{\boldsymbol{q}}_2(t) &= 0, \\
\boldsymbol{q}_{\mathrm{out}}(t) &= \boldsymbol{q}_1(t) + \boldsymbol{q}_{\mathrm{in}}(t), \\
\boldsymbol{p}_{\mathrm{out}}(t) &= \boldsymbol{p}_1(t) + \boldsymbol{p}_{\mathrm{in}}(t).
\end{aligned}
$$

- A block diagram of this transformed system is given below:



- For this example, we find the following features:

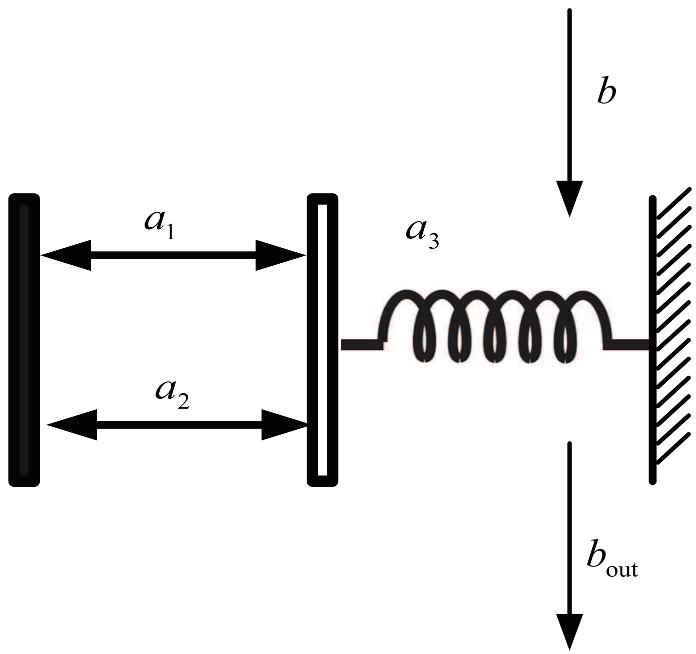
  (i) $\boldsymbol{p}_2$ is controllable but unobservable, while $\boldsymbol{q}_2$ is observable but uncontrollable. So, $\boldsymbol{q}_2$ is a quantum non-demolition (QND) variable.

Applications

We now apply the Kalman decomposition theory developed to two physical systems.

Example 2

- We investigate an opto-mechanical system, as shown below.



- The optical cavity has two optical modes, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. The cavity is coupled to a mechanical oscillator with mode $\boldsymbol{a}_3$, whose resonant frequency is ω_m.
- The coupling operator of the system is $\boldsymbol{L} = \sqrt{\kappa}\boldsymbol{a}_3$, where $\kappa > 0$.
- The Hamiltonian of the system is given by

$$\boldsymbol{H}_R = \omega_m(\boldsymbol{a}_1^*\boldsymbol{a}_1 + \boldsymbol{a}_2^*\boldsymbol{a}_2 + \boldsymbol{a}_3^*\boldsymbol{a}_3) + \frac{\lambda_1}{2}(\boldsymbol{a}_1\boldsymbol{a}_3^* + \boldsymbol{a}_1^*\boldsymbol{a}_3) + \frac{\lambda_2}{2}(\boldsymbol{a}_2\boldsymbol{a}_3^* + \boldsymbol{a}_2^*\boldsymbol{a}_3)$$

where $\lambda_1, \lambda_2 > 0$ are the opto-mechanical couplings.

Red-detuned case

- In this case, $\Delta_1 = \Delta_2 = -\omega_m < 0$.
- The coordinate transformation

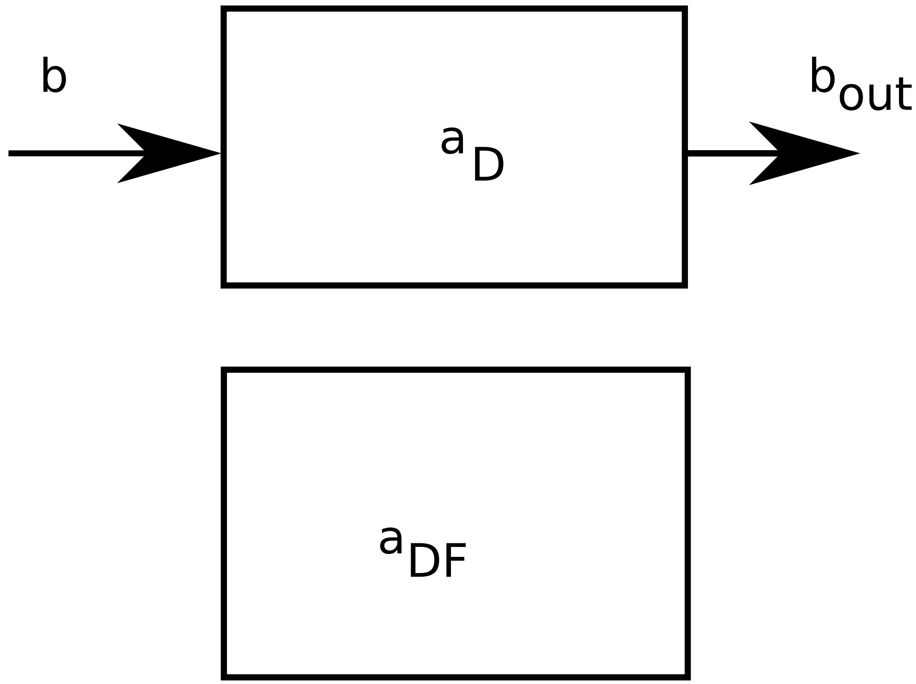
$$\left[\begin{array}{c} \boldsymbol{a}_{DF} \\ \hline \boldsymbol{a}_D \end{array}\right] = \left[\begin{array}{c} \rho_2 \boldsymbol{a}_1 - \rho_1 \boldsymbol{a}_2 \\ \hline \rho_1 \boldsymbol{a}_1 + \rho_2 \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{array}\right],$$

yields the following Kalman decomposition:

$$\begin{aligned} \dot{\boldsymbol{a}}_{DF}(t) &= -\imath\omega_m \boldsymbol{a}_{DF}(t), \\ \dot{\boldsymbol{a}}_D(t) &= -\begin{bmatrix} \imath\omega_m & \imath\frac{\lambda}{2} \\ \imath\frac{\lambda}{2} & \frac{\kappa}{2} + \imath\omega_m \end{bmatrix} \boldsymbol{a}_D(t) \\ &\quad - \begin{bmatrix} 0 \\ \sqrt{\kappa} \end{bmatrix} \boldsymbol{b}(t), \\ \boldsymbol{b}_{\mathrm{out}}(t) &= \begin{bmatrix} 0 & \sqrt{\kappa} \end{bmatrix} \boldsymbol{a}_D(t) + \boldsymbol{b}(t) \end{aligned}$$

where $\lambda = \sqrt{\lambda_1^2 + \lambda_2^2}$, $\rho_1 = \lambda_1/\lambda$, $\rho_2 = \lambda_2/\lambda$.

- A block diagram corresponding to this system is shown below.



- Clearly, $\boldsymbol{a}_{DF}$ is a DF mode.
- It is a linear combination of the two cavity modes and is decoupled from the mechanical mode, thus being immune from the mechanical damping. This mode is been called “mechanically dark”.
- In the real quadrature operator representation, the DF mode is

$$V_1 \begin{bmatrix} \boldsymbol{a}_{DF} \\ \boldsymbol{a}_{DF}^* \end{bmatrix} = \begin{bmatrix} \rho_2 \boldsymbol{q}_1 - \rho_1 \boldsymbol{q}_2 \\ \rho_2 \boldsymbol{p}_1 - \rho_1 \boldsymbol{p}_2 \end{bmatrix}.$$

Blue-detuned case

- In this case, the detuning between the laser frequency and both cavity modes is positive.
- Moreover, we assume $\Delta_1 = \Delta_2 = \omega_m > 0$.
- Under the rotating frame approximation $\boldsymbol{a}_1(t) \to \boldsymbol{a}_1(t)e^{-\imath\omega_m t}$, $\boldsymbol{a}_2(t) \to \boldsymbol{a}_2(t)e^{-\imath\omega_m t}$, and $\boldsymbol{a}_3(t) \to \boldsymbol{a}_3(t)e^{\imath\omega_m t}$, the Hamiltonian can be approximated by

$$\begin{aligned}\boldsymbol{H}_B &= \lambda_1 \frac{\boldsymbol{a}_1\boldsymbol{a}_3 + \boldsymbol{a}_1^*\boldsymbol{a}_3^*}{2} + \lambda_2 \frac{\boldsymbol{a}_2\boldsymbol{a}_3 + \boldsymbol{a}_2^*\boldsymbol{a}_3^*}{2} \\ &\quad -\omega_m \boldsymbol{a}_1^*\boldsymbol{a}_1 - \omega_m \boldsymbol{a}_2^*\boldsymbol{a}_2 + \omega_m \boldsymbol{a}_3^*\boldsymbol{a}_3.\end{aligned}$$

- In this case, we find that there are no $\bar{c}o$ or $c\bar{o}$ subsystems.
- By applying our quantum Kalman decomposition for the case of a real quadrature representation, we obtain the transormed variables

$$\boldsymbol{x}_{co} = \begin{bmatrix} \boldsymbol{q}_3 \\ \rho_1 \boldsymbol{q}_1 + \rho_2 \boldsymbol{q}_2 \\ \boldsymbol{p}_3 \\ \rho_1 \boldsymbol{p}_1 + \rho_2 \boldsymbol{p}_2 \end{bmatrix}, \quad \boldsymbol{x}_{\bar{c}\bar{o}} = \begin{bmatrix} \rho_2 \boldsymbol{q}_1 - \rho_1 \boldsymbol{q}_2 \\ \rho_2 \boldsymbol{p}_1 - \rho_1 \boldsymbol{p}_2 \end{bmatrix}.$$

- Also, the transformed system in Kalman Canonical form is

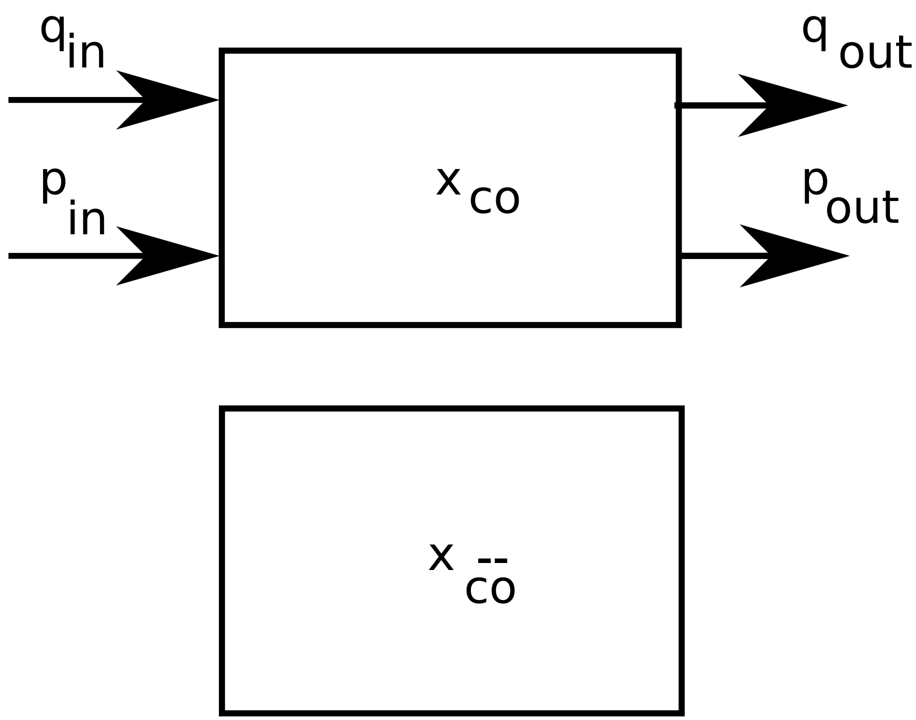
$$\begin{aligned}
\dot{\boldsymbol{x}}_{co}(t) &= A_{co}\boldsymbol{x}_{co}(t) - \begin{bmatrix} \sqrt{\kappa} & 0 \\ 0 & 0 \\ 0 & \sqrt{\kappa} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathrm{in}}(t) \\ \boldsymbol{p}_{\mathrm{in}}(t) \end{bmatrix}, \\
\dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) &= \begin{bmatrix} 0 & -\omega_m \\ \omega_m & 0 \end{bmatrix} \boldsymbol{x}_{\bar{c}\bar{o}}(t), \\
\begin{bmatrix} \boldsymbol{q}_{\mathrm{out}}(t) \\ \boldsymbol{p}_{\mathrm{out}}(t) \end{bmatrix} &= \begin{bmatrix} \sqrt{\kappa} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\kappa} & 0 \end{bmatrix} \boldsymbol{x}_{co}(t) + \begin{bmatrix} \boldsymbol{q}_{\mathrm{in}}(t) \\ \boldsymbol{p}_{\mathrm{in}}(t) \end{bmatrix},
\end{aligned}$$

where

$$A_{co} = \begin{bmatrix} -\kappa & 0 & \omega_m & -\frac{\lambda}{2} \\ 0 & 0 & -\frac{\lambda}{2} & -\omega_m \\ -\omega_m & -\frac{\lambda}{2} & -\kappa & 0 \\ -\frac{\lambda}{2} & \omega_m & 0 & 0 \end{bmatrix}.$$

- Clearly, $\boldsymbol{x}_{\bar{c}\bar{o}}$ is a DF mode. Indeed, it is exactly the same as that in the red-detuned regime case.

- A block diagram corresponding to this system is shown below.



Phase-shift regime

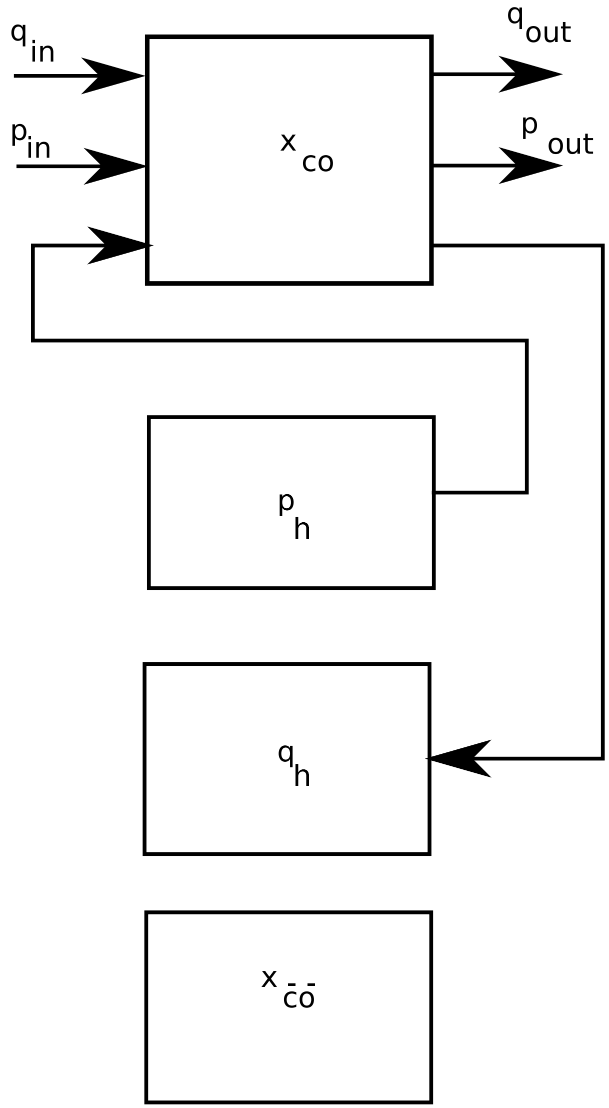
- In this case, the two cavity modes are resonant with their respective driving lasers.
- Moreover, $\Delta_1 = \Delta_2 = 0$.
- By applying our quantum Kalman decomposition for the case of a real quadrature representation, we obtain the transormed variables

$$\left[\begin{array}{c} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \\ \hline \boldsymbol{x}_{co}(t) \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}}(t) \end{array}\right] = \left[\begin{array}{c} -\rho_1 \boldsymbol{p}_1 - \rho_2 \boldsymbol{p}_2 \\ \rho_1 \boldsymbol{q}_1 + \rho_2 \boldsymbol{q}_2 \\ \hline \boldsymbol{q}_3 \\ \boldsymbol{p}_3 \\ \hline \rho_2 \boldsymbol{q}_1 - \rho_1 \boldsymbol{q}_2 \\ \rho_2 \boldsymbol{p}_1 - \rho_1 \boldsymbol{p}_2 \end{array}\right].$$

- Also, the transformed system in Kalman Canonical form is

$$\begin{aligned}
\begin{bmatrix} \dot{\boldsymbol{q}}_h(t) \\ \dot{\boldsymbol{p}}_h(t) \end{bmatrix} &= \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}_{co}(t), \\
\dot{\boldsymbol{x}}_{co}(t) &= \begin{bmatrix} -\kappa/2 & \omega_m \\ -\omega_m & -\kappa/2 \end{bmatrix} \boldsymbol{x}_{co}(t) \\
&\quad -\lambda \begin{bmatrix} 0 \\ \boldsymbol{p}_h(t) \end{bmatrix} - \sqrt{\kappa} \begin{bmatrix} \boldsymbol{q}_{\text{in}}(t) \\ \boldsymbol{p}_{\text{in}}(t) \end{bmatrix}, \\
\dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) &= 0, \\
\begin{bmatrix} \boldsymbol{q}_{\text{out}}(t) \\ \boldsymbol{p}_{\text{out}}(t) \end{bmatrix} &= \sqrt{\kappa} \boldsymbol{x}_{co}(t) + \begin{bmatrix} \boldsymbol{q}_{\text{in}}(t) \\ \boldsymbol{p}_{\text{in}}(t) \end{bmatrix}.
\end{aligned}$$

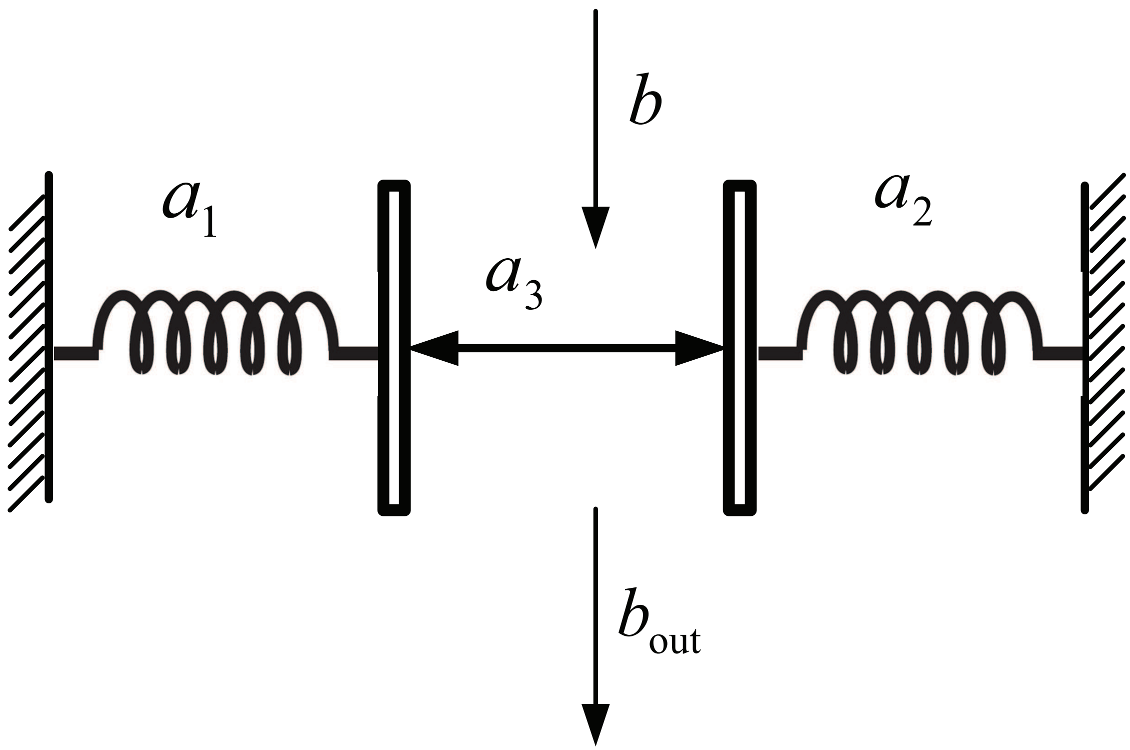
- A block diagram corresponding to this system is shown below.



- Clearly, $\boldsymbol{x}_{\bar{c}\bar{o}}(t)$ is a DF mode (which is the same as in the red-shift and blue-shift cases above.
- However, $\boldsymbol{p}_h(t)$ is a constant for all $t \geq 0$, thus is a QND variable.
- Actually, $\boldsymbol{p}_h$ could be measured continuously with no quantum limit on the predictability of these measurements as the measurement back-action only drives its conjugate operator $\boldsymbol{q}_h$.

Example 3

- The opto-mechanical system, as shown below has been studied theoretically and implemented experimentally using superconducting microwave circuits.



- Here, the two mechanical oscillators with modes $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, are not directly coupled.
- Instead, they are coupled to a microwave cavity, with mode $\boldsymbol{a}_3$.
- In this system, the mechanical damping is much smaller than the optical damping.
- Therefore, in what follows we neglect the mechanical damping.

- The system Hamiltonian is as follows:

$$\boldsymbol{H} = \Omega(\boldsymbol{a}_1^*\boldsymbol{a}_1 - \boldsymbol{a}_2^*\boldsymbol{a}_2) + g(\boldsymbol{a}_1 + \boldsymbol{a}_1^*)(\boldsymbol{a}_3 + \boldsymbol{a}_3^*) + g(\boldsymbol{a}_2 + \boldsymbol{a}_2^*)(\boldsymbol{a}_3 + \boldsymbol{a}_3^*).$$

- The optical coupling is $\boldsymbol{L} = \sqrt{\kappa}\boldsymbol{a}_3$.

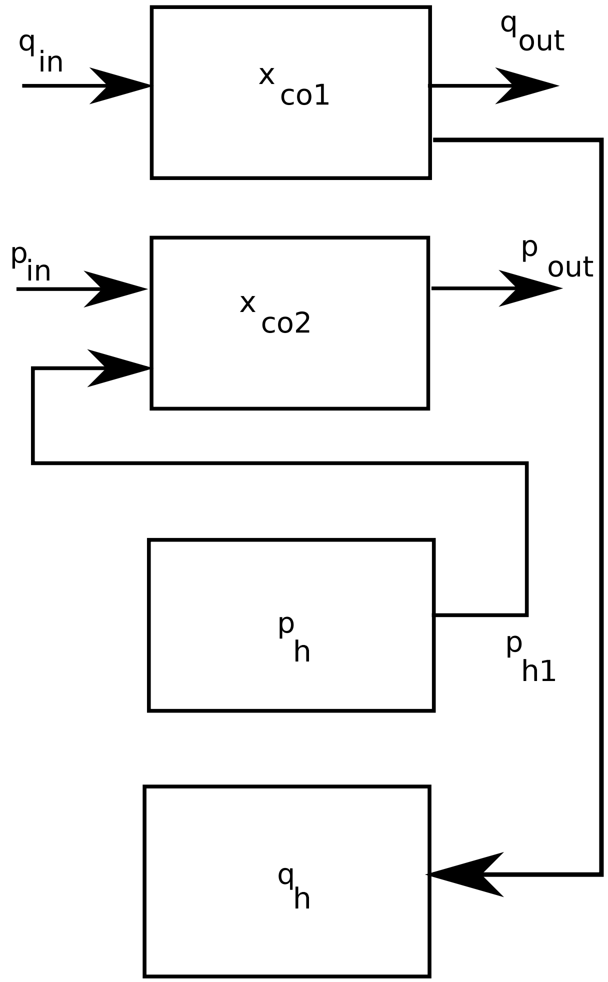
- By applying our quantum Kalman decomposition for the case of a real quadrature representation, we obtain the transormed variables

$$\left[\begin{array}{c} \boldsymbol{q}_h \\ \hline \boldsymbol{p}_h \end{array}\right] = \frac{1}{\sqrt{2}} \left[\begin{array}{c} \boldsymbol{q}_2 - \boldsymbol{q}_1 \\ -\boldsymbol{p}_1 - \boldsymbol{p}_2 \\ \hline \boldsymbol{p}_2 - \boldsymbol{p}_1 \\ \boldsymbol{q}_1 + \boldsymbol{q}_2 \end{array}\right], \quad \boldsymbol{x}_{co} = \begin{bmatrix} \boldsymbol{q}_3 \\ \boldsymbol{p}_3 \end{bmatrix}.$$

- Also, the transformed system in Kalman Canonical form is

$$\begin{aligned}
\dot{\boldsymbol{q}}_h(t) &= \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \boldsymbol{q}_h(t) + \begin{bmatrix} 0 & 0 \\ 2\sqrt{2}g & 0 \end{bmatrix} \boldsymbol{x}_{co}(t), \\
\dot{\boldsymbol{x}}_{co}(t) &= -\frac{\kappa}{2}\boldsymbol{x}_{co}(t) - \begin{bmatrix} 0 & 0 \\ 0 & 2\sqrt{2}g \end{bmatrix} \boldsymbol{p}_h(t) - \sqrt{\kappa} \begin{bmatrix} \boldsymbol{q}_{\text{in}}(t) \\ \boldsymbol{p}_{\text{in}}(t) \end{bmatrix}, \\
\dot{\boldsymbol{p}}_h(t) &= \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix} \boldsymbol{p}_h(t), \\
\begin{bmatrix} \boldsymbol{q}_{\text{out}}(t) \\ \boldsymbol{p}_{\text{out}}(t) \end{bmatrix} &= \sqrt{\kappa}\boldsymbol{x}_{co}(t) + \begin{bmatrix} \boldsymbol{q}_{\text{in}}(t) \\ \boldsymbol{p}_{\text{in}}(t) \end{bmatrix}.
\end{aligned}$$

- A block diagram corresponding to this system is shown below.



- The components of $\boldsymbol{p}_h$ are linear combinations of variables of the two mechanical oscillators, are immune from optical damping, and form a QMFS.
- Moreover, the second entry of $\boldsymbol{p}_h$, can be measured via a measurement on the optical cavity, and the back-action will only affect the dynamics of the mechanical quadratures in $\boldsymbol{q}_h$, which are conjugate to those in $\boldsymbol{p}_h$.
- It can be readily shown that the system realizes a BAE measurement of $\boldsymbol{q}_{\rm out}$ with respect to $\boldsymbol{p}_{\rm in}$, and a BAE measurement of $\boldsymbol{p}_{\rm out}$ with respect to $\boldsymbol{q}_{\rm in}$.
- Finally, notice that $\frac{\boldsymbol{q}_1+\boldsymbol{q}_2}{\sqrt{2}}$, the second entry of $\boldsymbol{p}_h$, couples to the microwave cavity dynamics $\boldsymbol{x}_{co}$.

Conclusions

- In this paper, we have studied the Kalman decomposition for linear quantum systems.
- We have shown that it can always be performed with an orthogonal symplectic coordinate transformation in the real quadrature representation. These are the only coordinate transformations allowed by quantum mechanics to preserve the physical realizability conditions for linear quantum systems.
- Because the coordinate transformations are orthogonal, they can be performed in a numerically stable way.
- Furthermore, the decomposition is performed in a constructive way, as in the classical case.
- The Kalman canonical decomposition naturally exposes the system's decoherence-free modes, quantum-nondemolition variables, quantum-mechanics-free-subspaces, which are important resources in quantum information science.