Robust Event-Triggered Control

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Outline

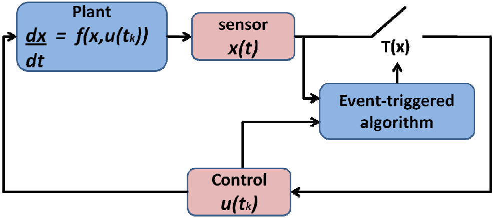
Objective

Given a class of uncertain linear plant and stabilizing controller, design of event-triggered rules to stabilize the closed-loop system. [Postoyan et al., 2015, Tallapragada and Chopra, 2012]

- The event-triggering rule depends only on local information, that is it uses only the output signals available to the controller [Tarbouriech et al., 2016], [Abdelrahim et al., 2014].
- The approach proposed combines a hybrid framework [Goebel et al., 2012] to describe the closed-loop system with looped functionals based techniques [Seuret, 2012, Briat and Seuret, 2012].

Problems

(1) How to design flow and jump conditions, defining the event-triggering rule, so that the obtained hybrid system is globally asymptotically stable? (2) What about the inter-event time? (3) Can we compute a maximal sampling period T?



- This problem corresponds to an emulation problem [Heemels et al., 2012, Wang and Lemmon, 2008, Postoyan et al., 2011, Tallapragada and Chopra, 2012].

- Consider a linear system fed by an output feedback sampled-data control given by the hybrid dynamical system

$$\left\{\begin{array}{lcl} \dot{x} & = & Ax + Bu, \\ \dot{u} & = & 0, \\ \dot{\sigma} & \in & g_T(\sigma), \end{array}\right. \qquad (x, u, \sigma) \in \mathcal{C},$$

$$\left\{\begin{array}{lcl} x^+ & = & x, \\ u^+ & = & KCx, \\ \sigma^+ & = & 0, \end{array}\right. \qquad (x, u, \sigma) \in \mathcal{D}, \tag{1}$$

where
  - $x \in \mathbb{R}^n$ represents the state of the system
  - $u \in \mathbb{R}^m$ represents the zero order holder of the system input since the last sampling time.
- The output of the system y is given by

$$y = Cx \in \mathbb{R}^p. \tag{2}$$

Remarks

- Such a system (1)-(2) can appear when connecting, for instance, a linear continuous plant with a dynamic output feedback controller.
  - ▷ Matrices A, B, C characterize the system dynamics and matrix K corresponds to the controller gain.
- Uncertainties. While C is assumed to be constant and known, let us assume that matrices A and B are constant but uncertain, such that

$$[A \; B] \quad \in \quad Co\{\; [A_i \; B_i] \;\}_{i \in \mathcal{I}}, \tag{3}$$

for some constant and known matrices A_i and B_i, for $i \in \mathcal{I}$ where $\mathcal{I}$ is a bounded subspace of $\mathbb{N}$.

Remarks

- Timer $\sigma \in [0, 2T]$ flows by keeping track of the elapsed time since the last sample (where it was reset to zero) according to the following set-valued dynamics:

$$g_T(\sigma) \quad := \quad \begin{cases} 1 & \sigma \leq 2T \\ [0,1] & \sigma = 2T, \end{cases} \tag{4}$$

  - ▷ whenever $\sigma < 2T$, its value exactly represents the elapsed time since the last sample,
  - ▷ moreover $\sigma \in [T, 2T]$ implies that at least T seconds have elapsed since the last sample.

Problem

Given an uncertain linear plant and a hybrid controller defined by matrices A_i, B_i for $i \in \mathcal{I}$ and K, C. Design an event-triggering rule, with a prescribed dwell-time T

▷ That is the flow set $\mathcal{C}$, the jump set $\mathcal{D}$ and T

that makes the closed-loop system (1)-(4) globally asymptotically stable to a compact set wherein $x = 0$ and $u = 0$.

- The role of the flow and jump sets $\mathcal{C}$ and $\mathcal{D}$ is to rule when a sampling should happen, based on the available signals to the controller, namely output $y = Cx$, the last sampled input u and timer σ.

- We select the following sets $\mathcal{C}$ and $\mathcal{D}$:

$$\begin{aligned} \mathcal{C} &:= \mathcal{F} \cup \{\sigma \in [0, T]\} \quad (5a) \\ \mathcal{D} &:= \mathcal{J} \cap \{\sigma \in [T, 2T]\}, \quad (5b) \end{aligned}$$

where sets $\mathcal{F}$ and $\mathcal{J}$ are selected as

$$\mathcal{F} := \left\{ (x, u) : \begin{bmatrix} y \\ s - Ky \end{bmatrix}^\top M \begin{bmatrix} y \\ s - Ky \end{bmatrix} \leq 0 \right\}, \quad (5c)$$

$$\mathcal{J} := \left\{ (x, u) : \begin{bmatrix} y \\ s - Ky \end{bmatrix}^\top M \begin{bmatrix} y \\ s - Ky \end{bmatrix} \geq 0 \right\}, \quad (5d)$$

- Matrix $M = \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \in \mathbb{R}^{(p+m)\times(p+m)}$ has to be designed.
- The considered event-triggered problem is parametrized by M and T.

Remarks

- The jump set selection in (5b) ensures that all solutions satisfy a dwell-time constraint corresponding to T.
    - ▷ Indeed, jumps are inhibited unless timer $\sigma \geq T$, which implies that at least T ordinary time elapses between each pair of consecutive sampling times.
- Selecting $M_2 = 0$ leads to the definition of the flow and jump sets usually employed in the literature, issued from an Input-to-State (or Input-to-Output) analysis. See [Postoyan and Girard, 2015] for more details.

Lyapunov function

- The proof of our theorems is based on the use of a non-smooth Lyapunov function and LaSalle principle.
- In particular, we use the following function:[1]

$$V(x,u,\sigma) := \underbrace{\mathrm{e}^{-\rho \min\{\sigma,T\}} \left| \Lambda(T - \min\{\sigma, T\}) \begin{bmatrix} x \\ u \end{bmatrix} \right|_P^2}_{=:V_0(x,u,\sigma)} + \underbrace{\eta |u|^2}_{=:V_u(u)}, \tag{6}$$

with Λ given by

$$\Lambda(A,B,T) := \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T} \in \mathbb{R}^{n \times 2n}. \tag{7}$$

and where ρ and η are sufficiently small positive scalars to be selected.

[1] Here we use the standard notation $|z|_P^2 := z^\top P z$.

Theorem 1 (matrices A, B are constant and known)

Assume that there exist matrices $P \in \mathbb{S}^n$, $M = \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \in \mathbb{S}^{p+m}$ satisfying

$$\begin{aligned} \Psi_M(A,B) &:= \begin{bmatrix} \mathrm{He}(PA_{cl}) - C^\top M_1 C & PB - C^\top M_2 \\ B^\top P - M_2^\top C & -M_3 \end{bmatrix} < 0, \\ \Phi(A,B,T) &:= \left(\Lambda(A,B,T)\begin{bmatrix} I \\ KC \end{bmatrix}\right)^\top P \Lambda(A,B,T)\begin{bmatrix} I \\ KC \end{bmatrix} - P < 0, \end{aligned} \tag{8}$$

with $A_{cl} := A + BKC$ and $\Lambda(A,B,T)$ defined in (7). Then the compact attractor

$$\mathcal{A} := \{(x,u,\sigma) :\ x = 0, u = 0, \sigma \in [0, 2T]\}, \tag{9}$$

is GAS for the nominal closed-loop dynamics (1), (5).

Remarks

- The LMI conditions can be interpreted as follows
  - ▷ The condition $\Psi_M(A, B) < 0$ imposes that the Lyapunov function V in (6) is decreasing while flowing with $\sigma \geq T$ (which requires $(x, u) \in \mathcal{F}$).
  - ▷ The condition $\Phi(A, B, T) < 0$ guarantees that the Lyapunov function V in (6) is non-increasing while flowing and when $\sigma < T$.
  - ▷ The condition $\Phi(A, B, T) < 0$ can be interpreted as an asymptotic stability criterion for system (1) when the control updates are performed periodically with a period T, which motivates the union and intersection in (5a) and (5b).
  - ▷ The dwell time T appears as a parameter for the design of event trigger algorithm.

- When matrices A, B and parameter T are known and constant, inequality $\Phi(A, B, T) < 0$ can be easily implemented and verified.
- When matrices A and B are uncertain, verifying inequality $\Phi(A, B, T) < 0$ for any pair (A, B) in (3) becomes a difficult nonlinear problem.
  - ▷ Now we propose a method to deal with uncertain matrices A and B based from [Seuret, 2012, Thm 1] and recent developments arising from stability analysis of persistent sampled-data systems [Hetel et al., 2017].

Theorem 2 (uncertain case)

Assume that there exist matrices $P \in \mathbb{S}^n$, $M := \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \in \mathbb{S}^{p+m}$, and matrices $Z \in \mathbb{S}^n_+$, $Q, U \in \mathbb{S}^n$, $R \in \mathbb{R}^{n\times n}$ and $Y_i \in \mathbb{R}^{2n\times n}$, $i = 1, \ldots, m$ satisfying conditions $\Psi_M(A_i, B_i) < 0$, given in (8) and

$$\begin{array}{rcl} \Theta_1(A_i,B_i,T) := & F_0(A_i,B_i,T) + TF_1(A_i,B_i) & < 0, \\ \Theta_2(A_i,B_i,T) := & \begin{bmatrix} F_0(A_i,B_i,T) & TY_i \\ \star & -TZ \end{bmatrix} & < 0, \end{array} \tag{10}$$

hold for all $i = 1, \ldots, m$ with

$$\begin{array}{l} F_0(A_i,B_i,T) := T\mathrm{He}\{e_{0i}^\top P e_1 - Y_i e_{12} - e_{12}^\top R e_2\} - e_{12}^\top Q e_{12} - e_2^\top T X e_2, \\ F_1(A_i,B_i) := \mathrm{He}[e_{0i}^\top Q e_{12} + e_{0i}^\top R e_2] + e_{0i}^\top Z e_{0i} + 2 e_2^\top X e_2, \end{array}$$

and $e_{0i} := \begin{bmatrix} A_i & B_iKC \end{bmatrix}$, $e_1 := \begin{bmatrix} I_n & 0 \end{bmatrix}$, $e_2 := \begin{bmatrix} 0 & I_n \end{bmatrix}$, $e_{12} := \begin{bmatrix} I_n & -I_n \end{bmatrix}$. Then the compact attractor $\mathcal{A}$ in (9) is GAS for the uncertain closed-loop dynamics (1)-(4), (5).

- One shows that function V in (6) is a non-strict Lyapunov function for the nominal and uncertain closed loops.

$$V(x,u,\sigma) := \underbrace{e^{-\rho\min\{\sigma,T\}}\left|\Lambda(T-\min\{\sigma,T\})\begin{bmatrix}x\\u\end{bmatrix}\right|_P^2}_{=:V_0(x,u,\sigma)} + \underbrace{\eta|u|^2}_{=:V_u(u)},$$

- Along flowing solutions we obtain:

$$\dot{V}(\xi) \leq -\varepsilon\left|\begin{bmatrix}x\\u-Ky\end{bmatrix}\right|^2, \text{ if } (x,u)\in\mathcal{F}, \sigma \geq T. \quad (11)$$

- For all $\xi \in \mathcal{D}$,

$$V^+(\xi) = e^{-\rho T} V_0(\xi) \leq e^{-\rho T} V(\xi) \quad (12)$$

  which proves the strict decrease of the Lyapunov function, across any jump outside $\mathcal{A}$.

- No "bad" complete solution exists, which keeps V constant and nonzero. If any such "bad" complete solution exists, then it has to start outside $\mathcal{A}$ and it cannot jump because otherwise from (12).

- We propose an extension of La Salle's invariance principle based on the invariance principle in [Sanfelice et al., 2007] and [Goebel et al., 2012, Ch. 8] and some observations (already made in [Goebel et al., 2009]).
  - ▷ There is no need to check the flow and jump conditions in the attractor, that the flow condition only needs to be checked in the directions of the tangent cone to the flow set (as already established in [Sanfelice et al., 2007, Thm. 4.7]),
  - ▷ Nonsmooth Lyapunov functions V only need to be locally Lipschitz in the flow set and continuous in the jump set, and then rely on Clarke's generalized gradient [Clarke, 1990] for dealing with flowing solutions.

- A natural optimization procedure consists in the minimization of the effect of the off-diagonal term $PB_i - C^\top M_2$.
  - ▷ This optimization can be performed by minimizing M_3.
- This optimization problem is an LMI optimization as follows

$$\begin{array}{l} \min_{P,M} \; Tr(M_3), \text{ subject to: } \quad P > I, \; M_1 < 0, \\ \left.\begin{array}{rcl} \Psi_M(A_i, B_i) & < & 0, \\ \Theta_j(A_i, B_i, T) & < & 0, \; j = 1, 2, \end{array}\right. \quad \forall i \in \mathcal{I}. \end{array} \tag{13}$$

- $P > I$ has been imposed for well conditioning the LMI constraints.
- $M_1 < 0$ has been included in order to obtain $\text{He}(PA_{cli}) < 0$ in (8), which avoids exponentially unstable continuous dynamics, thereby giving more graceful inter-sample transients.
- Minimizing $Tr(M_3)$, increases the negativity of M_3 and leads to larger flow sets ((5)). Since the jump set is the closed complement of the flow set, it is expected that solutions will flow longer and jump less in light of larger flow sets.

- The plant [Donkers and Heemels, 2012, Abdelrahim et al., 2014] is

$$\begin{cases} \dot{x}_p = A_p(\omega)x_p + B_p(\omega)u_p, \\ y_p = C_p x_p, \end{cases} \tag{14}$$

  - ▷ with matrices

$$A_p(\omega) := \begin{bmatrix} 0 & 1 \\ -2 & 3+\omega \end{bmatrix}, \ B_p(\omega) := \begin{bmatrix} 0 \\ 1+0.1\omega \end{bmatrix}, \ C_p^\top := \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

  - ▷ $\omega \in \Omega := [-\omega_0, \omega_0]$ represents a constant uncertainty affecting the system for some positive constant ω_0.

- The dynamic output feedback controller is

$$\begin{cases} \dot{x}_c = A_c x_c + B_c y_p, \\ u_p = C_c x_c + D_c y_p, \end{cases} \tag{15}$$

with $A_c := \begin{bmatrix} 1.0919 & -1.1422 \\ 4.9734 & -6.1425 \end{bmatrix}, \quad B_c := \begin{bmatrix} 16.7501 \\ 64.6472 \end{bmatrix},$

$C_c := \begin{bmatrix} 0.1157 & -0.0928 \end{bmatrix}, \quad D_c := 0.$

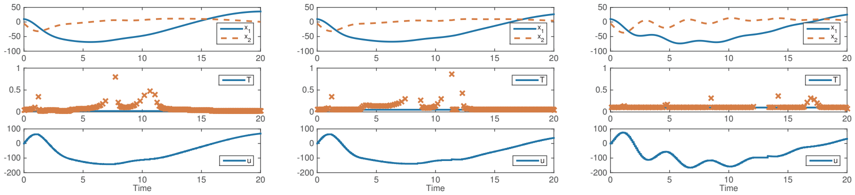
- Denote

$$x := \begin{bmatrix} x_p \\ x_c \end{bmatrix}$$

- The whole dynamics described by (14) and (15) can be reformulated as system (1) with

$$\left[\begin{array}{c|c} A(\omega) & B(\omega) \\ \hline K & C \end{array}\right] \in \left\{ \left[\begin{array}{cc|cc} A_p(\omega) & 0 & B_p(\omega) & 0 \\ 0 & A_c & 0 & B_c \\ \hline D_c & C_c & C_p & 0 \\ I & 0 & 0 & I \end{array}\right], \omega \in \Omega \right\}. \tag{16}$$

- $\omega_0 = 0$
- in [Abdelrahim et al., 2014] a dwell-time $T = 0.0114s$ is obtained.
- With our approach solutions to the conditions of Theorem 1 one obtaints T up to $0.11s$, which is ten times larger than the solution provided in [Abdelrahim et al., 2014].
  - ▷ This demonstrates the potential of the proposed method.



$N_u = 434$ $N_u = 255$ $N_u = 165$

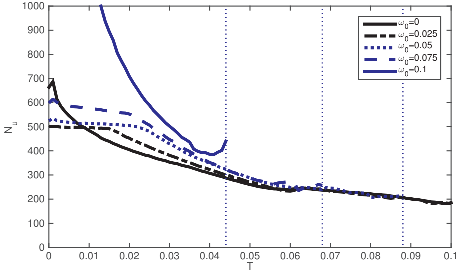
- Simulations obtained for $T = 0.02, 0.05$ and $0.10s$ and where matrix M results from the optimization problem (13), with $x_{p0} = [10 \ \ -5]^\top$, $x_{c0} = [0 \ 0]^\top$ and $\sigma = 0$.
- Classical trade-off between the number of control updates and the performance of the closed-loop system.

- $\omega_0 \neq 0$

ω_0	0	0.04	0.08	0.12	0.139
Th.1	0.114	–	–	–	–
Th.2	0.112	0.100	0.070	0.028	0.008

Table: Maximal dwell time T_{max} leading to feasibility of the conditions in Theorems 1 and 2 for several values of ω_0.

- Evolution of N_u w.r.t T for several values of ω_0.
- The vertical dotted lines represent the limit values of the T for which the conditions of Theorem 2 are feasible for $\omega_0 = 0.1$ (left), $\omega_0 = 0.075$ (middle) and $\omega_0 = 0.05$ (right).



- We provided a way to design the event-triggered rules for uncertain linear systems controlled by means of a dynamic output feedback controller.
- A nonstrict and nonsmooth Lyapunov functions has been used.
- Numerically tractable conditions allow to guarantee an adjustable dwell time of the solutions.
- Future work. Address the co-design problem: to simultaneously design the feedback stabilizer and its event-triggered sampled data implementation.

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